Topic 0: Vector spaces

0.1 Basic notation

Here are some of the fundamental sets and spaces that we shall use throughout these notes.

- The set of natural numbers, that is, $\{1, 2, 3, \ldots\}$, is denoted $\mathbb{N}$, the set of rational numbers is denoted $\mathbb{Q}$, and the set of real numbers is denoted $\boldsymbol{R}$.
- The set of extended real numbers, that is, $\boldsymbol{R} \cup \{\infty, -\infty\}$, is denoted $\boldsymbol{R}^\sharp$.
- The m-dimensional the vector space of ordered lists of m real numbers is denoted $\boldsymbol{R}^{\mathrm{m}}$. Similarly, $\boldsymbol{R}^{\mathbb{N}}$ denotes the set of all sequences of real numbers.
- The space ℓ_p of ***p*-summable sequences** is defined by

$$\ell_p = \Big\{(x_1, x_2, \ldots) \in \boldsymbol{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x_k|^p < \infty\Big\}.$$

- The **Euclidean inner product** on $\boldsymbol{R}^{\mathrm{m}}$ or ℓ_2 is denoted $x \cdot y$ and is given by

$$x \cdot y = \sum_{j=1}^{m} x_j y_j$$

for $\boldsymbol{R}^{\mathrm{m}}$ and $\sum_{j=1}^{\infty} x_j y_j$ for ℓ_2.

The Euclidean inner product gives rise to the **Euclidean norm**

$$\|x\| = \sqrt{x \cdot x},$$

which is $\left(\sum_{k=1}^{m} x_k^2\right)^{1/2}$ for $\boldsymbol{R}^{\mathrm{m}}$.

In general, the norm on ℓ_p (where $p \geqslant 1$) is given by $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$. The space of bounded sequences,

$$\ell_\infty = \Big\{(x_1, x_2, \ldots) \in \boldsymbol{R}^{\mathbb{N}} : \sup_k x_k < \infty\Big\},$$

is given the norm

$$\|x\|_\infty = \sup_k |x_k|.$$

I adopt the convention of denoting the i^{th} **unit coordinate vector** by e^i, regardless of the dimension of the underlying space. That is, the i^{th} coordinate of e^i is one and all the others are zero. Similarly, the **unit vector**, which has all its components equal to one, is denoted **1**, regardless of the dimension of the space. (This includes infinite-dimensional sequence spaces.)

Throughout these notes I adopt David Gale's [1] notational convention which does not distinguish between row and column vectors. This means that if A is an $m \times n$ matrix, and x is a vector, and I write Ax, you infer that x is an n-dimensional column vector, and if I write yA, you infer that y is an m-dimensional row vector. Similarly, I could write xy instead of $x \cdot y$. The notation yAx means that x is an n-dimensional column vector, y is an m-dimensional row vector, and yAx is the scalar $yA \cdot x = y \cdot Ax$.

0.1.1 $\boldsymbol{R}^{\mathrm{m}}$ is an ordered vector space

The usual order relation on the set $\boldsymbol{R}$ of real numbers is denoted $\geqslant$ ($x \geqslant y$ means that x is greater than or equal to y) or $\leqslant$ ($x \leqslant y$ means that y is greater than or equal to x). On the vector space $\boldsymbol{R}^{\mathrm{m}}$, the set of ordered m-tuples of reals, we have the following partial orders.

$$\begin{aligned} x \geqq y &\iff x_i \geqslant y_i,\ i = 1, \ldots, m \\ x > y &\iff x_i \geqslant y_i,\ i = 1, \ldots, m, \text{ and } x \neq y \\ x \gg y &\iff x_i > y_i,\ i = 1, \ldots, m. \end{aligned}$$

The notations $x \leqq y$, etc., are defined in a similar fashion. We say that a vector

- x is **nonnegative** if $x \geqq 0$.
- x is **semipositive** if $x > 0$.
- x is **strictly positive** if $x \gg 0$.

Finally,

- $\boldsymbol{R}^{\mathrm{m}}_{+} = \{x \in \boldsymbol{R}^{\mathrm{m}} : x \geqq 0\}$ is the **nonnegative orthant** of $\boldsymbol{R}^{\mathrm{m}}$, and
- $\boldsymbol{R}^{\mathrm{m}}_{++} = \{x \in \boldsymbol{R}^{\mathrm{m}} : x \gg 0\}$ is the **strictly positive orthant** of $\boldsymbol{R}^{\mathrm{m}}$.

I shall try to avoid using the adjective "positive" by itself, since to most mathematicians it means "nonnegative," but to many non-mathematicians it means "strictly positive."

Let me call your attention to the following fact about nonnegative vectors. While the result is simple, and almost self-evident, it is used over and over again, so it is worth giving it a name.

0.1.1 The Nonnegativity Test *For $p \in \boldsymbol{R}^{\mathrm{m}}$, the following statements are equivalent:*

1. $p \geqq 0$.
2. $(\forall x \geqq 0)\ [\,p \cdot x \geqslant 0\,]$.
3. $(\exists \alpha \in \boldsymbol{R})\ (\forall x \geqq 0)\ [\,p \cdot x \geqslant \alpha\,]$.

Similarly, these statements are equivalent:

1′. $p \leqq 0$.

2′. $(\forall x \geqq 0)\ [\,p \cdot x \leqslant 0\,]$.

3′. $(\exists \alpha \in \boldsymbol{R})\ (\forall x \geqq 0)\ [\,p \cdot x \leqslant \alpha\,]$.

Proof: Clearly (1) $\implies$ (2) $\implies$ (3). To see that (3) $\implies$ (1), consider x of the form $x = \lambda e^i$ where $\lambda > 0$. Then $p \cdot x = \lambda p_i$, so by (3) we have that $\lambda p_i \geqslant \alpha$ for every $\lambda > 0$. Dividing by $\lambda > 0$ gives $p_i \geqslant \alpha/\lambda$. Letting $\lambda \to \infty$ yields $p_i \geqslant 0$. The equivalence of the primed statements is proven similarly. ∎

0.2 Some geometry of vector spaces

0.2.1 Sum of sets

We can think of vectors as being added "tip-to-tail." (See Figure 0.2.1.) This lets us visualize the sum of two sets of vectors.

0.2.1 Definition *Let A and B be sets in a vector space X. The* ***sum*** *of A and B is*

$$A + B = \{x + y \in X : x \in A,\ y \in B\}.$$

(See Figure 0.2.2.) This is sometimes called the ***Minkowski sum*** *of A and B.*

Note that for any set A,

$$A + \varnothing = \varnothing.$$

We also write the sum

$$A + y = \{x + y : x \in A\},$$

so that

$$A + B = \bigcup_{y \in B} A + y.$$

Note that $A + y = A + (y - x)$ for any $x \in A$.

Also, set addition is commutative and associative:

$$A + B = B + A, \qquad (A + B) + C = A + (B + C).$$

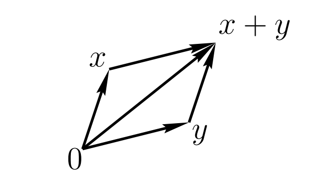

Figure 0.2.1. The sum of two vectors.

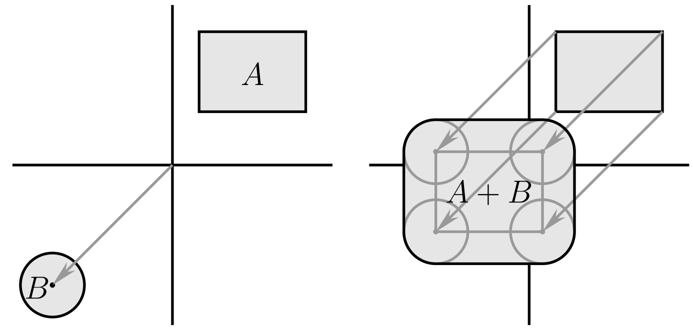

Figure 0.2.2. Sum of two sets.

0.2.2 Example Let

$$E = \{(x, y) \in \boldsymbol{R}^2 : y \geqslant 1/x \text{ and } x > 0\}$$

and

$$F = \{(x, y) \in \boldsymbol{R}^2 : y \geqslant -1/x \text{ and } x < 0\}.$$

See Figure 0.2.3. Note that while E and F are closed, their sum

$$E + F = \{(x, y) \in \boldsymbol{R}^2 : y > 0\}$$

is not closed. Topic 20 discusses conditions under which the sum of closed sets is closed.

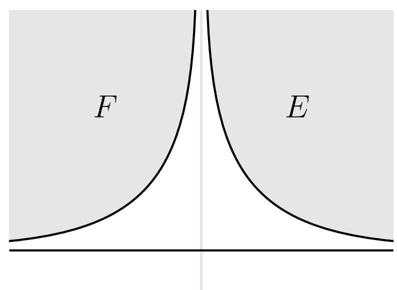

Figure 0.2.3.

□

0.2.3 Exercise For the following pairs of sets, sketch the sets and their sums.

1. $A = \{x \in \boldsymbol{R}^2 : \|x\| \leqslant 1\}$, $B = \{x \in \boldsymbol{R}^2 : \|x\| \leqslant 2\}$
2. $A = \{(\xi, 1-\xi) : 0 \leqslant \xi \leqslant 1\}$, $B = \{x \in \boldsymbol{R}^2 : \|x\| \leqslant 1/2\}$.
3. $A = \{(\xi, 1-\xi) : 0 \leqslant \xi \leqslant 1\}$, $B = \{(\xi, \eta) : 1/3 \leqslant \eta/\xi \leqslant 2/3,\ \xi \geqslant 0\}$.
4. $A = \{(\xi, \eta) : \xi \geqslant 0,\ \eta = \xi/3\}$, $B = \{(\xi, \eta) : \xi \geqslant 0,\ \eta = 2\xi/3\}$.

□

0.2.2 Scalar Multiples of sets

Elaborate

$$\alpha A = \{\alpha x : x \in A\}.$$

Warning: In general,

$$\underbrace{A + \cdots + A}_{n \text{ terms}} \neq nA$$

For instance, let $A = \{0, 1\} \subset \boldsymbol{R}$. Then $\underbrace{A + \cdots + A}_{n \text{ terms}} = \{0, 1, \ldots, n\}$ and $nA = \{0, n\}$.

0.2.3 Geometry of the dot product

To see that

$$x \cdot y = \|x\| \, \|y\| \, \cos\theta,$$

where θ is the angle between x and y, orthogonally project y on the space spanned by x. That is, write $y = \alpha x + z$ where $z \cdot x = 0$. Thus

$$z \cdot x = (y - \alpha x) \cdot x = y \cdot x - \alpha x \cdot x = 0 \quad \Longrightarrow \quad \alpha = x \cdot y / x \cdot x.$$

Referring to Figure 0.2.4 we see that

$$\cos\theta = \alpha \|x\| / \|y\| = x \cdot y / \|x\| \, \|y\|.$$

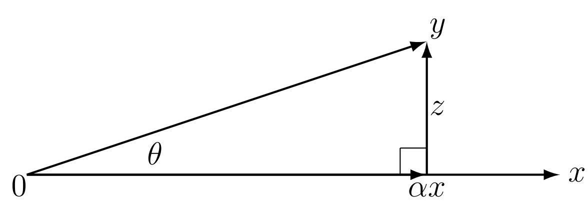

Figure 0.2.4. Dot product and angles

For a nonzero $p \in \boldsymbol{R}^{\mathrm{m}}$,

$$\{x \in \boldsymbol{R}^{\mathrm{m}} : p \cdot x = 0\}$$

is a linear subspace of dimension $m - 1$. It is the subspace of all vectors x making a right angle with p.

A set of the form

$$\{x \in \boldsymbol{R}^{\mathrm{m}} : p \cdot x = c\}, \qquad p \neq 0$$

is called a **hyperplane**. To visualize the hyperplane $H = \{x : p \cdot x = c\}$ start with the vector $\alpha p \in H$, where $\alpha = c/p \cdot p$. Draw a line perpendicular to p at the point αp. For any x on this line, consider the right triangle with vertices $0, (\alpha p), x$. The angle x makes with p has cosine equal to $\|\alpha p\| / \|x\|$, so $p \cdot x = \|p\| \|x\| \|\alpha p\| / \|x\| = \alpha p \cdot p = c$. That is, the line lies in the hyperplane H. See Figure 0.2.5.

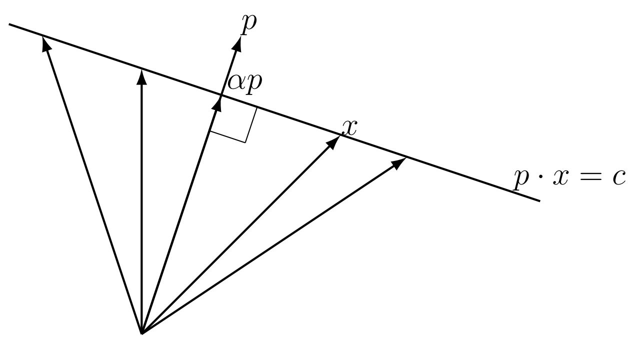

Figure 0.2.5. A hyperplane.

0.2.4 Lines, segments, and rays

In this section, and indeed throughout these notes the expression $(1-\lambda)x+\lambda y$ or something like it occurs so frequently that it is a good idea to have a short-hand name for it.

0.2.4 Definition *Given a vector space X, the* ***affine combination function*** *$\kappa\colon X \times X \times \boldsymbol{R} \to X$ is defined by*

$$\kappa(x,y,\lambda) = (1-\lambda)x + \lambda y = x + \lambda(y-x).$$

When X is a topological vector space (tvs) (see Definition A.11.1 in the appendix), then κ is continuous.

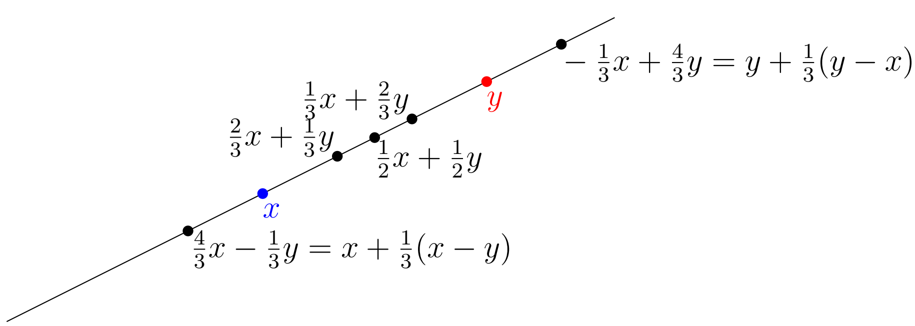

Figure 0.2.6. Points on the line through x and y.

0.2.5 Definition *Given two points x and y in a vector space, the* ***line segment*** *joining them, denoted $[x,y]$ is given by*

$$[x,y] = \{(1-\lambda)x + \lambda y : 0 \leqslant \lambda \leqslant 1\} = \{x + \lambda(y-x) : 0 \leqslant \lambda \leqslant 1\}.$$

We also write

$$\begin{aligned}[x,y) &= \{x+\lambda(y-x) : 0 \leqslant \lambda < 1\} = [x,y] \setminus \{y\},\\ (x,y] &= \{x+\lambda(y-x) : 0 < \lambda \leqslant 1\} = [x,y] \setminus \{x\}, \text{ and}\\ (x,y) &= \{x+\lambda(y-x) : 0 < \lambda < 1\} = [x,y] \setminus \{x,y\}.\end{aligned}$$

If $x \neq y$, they determine a unique ***line****, namely*

$$\{(1-\lambda)x + \lambda y : \lambda \in \boldsymbol{R}\} = \{x + \lambda(y-x) : \lambda \in \boldsymbol{R}\}.$$

Any nonzero point x determines a ***ray****, denoted $\langle x\rangle$, by*

$$\langle x\rangle = \{\lambda x : \lambda \geqslant 0\}.$$

A ***half-line*** *is the sum of a point and a ray, that is, a set of the form*

$$\{x + \lambda y : \lambda \geqslant 0\} = x + \langle y\rangle,$$

where $y \neq 0$.

0.3 Linear functions

Recall that a function f between vector spaces is **linear** if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Linear functions between vector spaces are often referred to as **linear transformations**. We treat $\boldsymbol{R}$ as a one-dimensional vector space over $\boldsymbol{R}$. Linear functions from a vector space to $\boldsymbol{R}$ are often called **linear functionals**, especially if the vector space is infinite dimensional. The set of linear functions from X to Y, denoted $\mathsf{L}(X,Y)$ is itself a vector space under the pointwise operations. The space $\mathsf{L}(X,\boldsymbol{R})$ is called the **dual space** of X, and is often denoted X^*.

A function $f\colon \boldsymbol{R}^{\mathrm{n}} \to \boldsymbol{R}^{\mathrm{m}}$ is linear if and only if there is some $m \times n$ matrix M such that $f(x) = Mx$. As such it must be continuous. In particular, $f\colon \boldsymbol{R}^{\mathrm{m}} \to \boldsymbol{R}$ is linear if and only if there exists some vector $p \in \boldsymbol{R}^{\mathrm{m}}$ such that $f(x) = p \cdot x$. (Let p be the vector whose i^{th} coordinate is $f(e^i)$.) Every linear function on $\boldsymbol{R}^{\mathrm{m}}$ is continuous. In other words, the dual space $\boldsymbol{R}^{\mathrm{m}*}$ of $\boldsymbol{R}^{\mathrm{m}}$ can be identified with $\boldsymbol{R}^{\mathrm{m}}$. This is a very special property of $\boldsymbol{R}^{\mathrm{m}}$ and a few infinite-dimensional vector spaces.

Most remarkable is that for infinite dimensional vector spaces there will be discontinuous linear functionals. This is most easily seen for normed spaces.

0.3.1 Lemma *Let X be a normed vector space, and let $U = \{x \in X : \|x\| \leqslant 1\}$ be its unit ball. If $f\colon C \to \boldsymbol{R}$ is linear, then f is continuous if and only if f is bounded on U.*

Proof: Since $x_n \to x$ if and only if $\|x_n - x\| \to 0$, it suffices consider continuity at 0. So assume f is continuous at 0. Then (taking $\varepsilon = 1$) there is a $\delta > 0$ such that if $\|x\| \leqslant \delta$, then $|f(x)| < 1$. So if $\|x\| \leqslant 1$, then $\|\delta x\| \leqslant \delta$, so $|f(\delta x)| < 1$, which implies $|f(x)| < 1/\delta$. That is, $|f(x)|$ is bounded by $1/\delta$ on U. The converse is similar. ∎

0.3.2 Proposition *Every infinite dimensional normed space has a discontinuous linear functional.*

Proof: If X is an infinite dimensional normed space, then it has an infinite Hamel basis B. We may normalize each basis vector to have norm one. Let $C = \{x_1, x_2, \dots\}$ be a countable subset of the basis B. Define the function ℓ on the basis B by $\ell(x_n) = n$ for $x_n \in C$, and $\ell(v) = 0$ for $v \in B \setminus C$. Every $y \in X$ has a unique representation as

$$y = \sum_{v \in B} \eta_v v,$$

where only finitely many η_v are nonzero. Extend ℓ from B to X by

$$\ell(y) = \sum_{v \in B} \eta_v \ell(v).$$

Then ℓ is a linear functional, but it is not bounded on the unit ball (as $\ell(x_n) = n$). So by Lemma 0.3.1 it is not continuous. ∎

0.4 Aside: The Summation Principle

The following lemma is trivial, but sufficiently useful that I have decided to give it a name.

0.4.1 Summation Principle *Let $A_1, \dots, A_n$ be nonempty subsets of a vector space X, and let $x_i \in A_i$ for $i = 1, \dots, n$. Let*

$$x = x_1 + \cdots + x_n.$$

If $p\colon X \to \boldsymbol{R}$ is a linear function, then

x maximizes p over $A_1 + \cdots + A_n$

if and only if

for each i, x_i maximizes p over A_i

The proof is a simple application of the definitions, and the fact that summation preserves inequalities. Note that we can replace maximization by minimization in the statement.

0.4.2 Exercise Write out a proof of the Summation Principle. □

Geometrically, maximizing p over a set Y amount to finding the "highest" hyperplane orthogonal to p that touches Y. See Figure 0.4.1.

Draw a better picture.

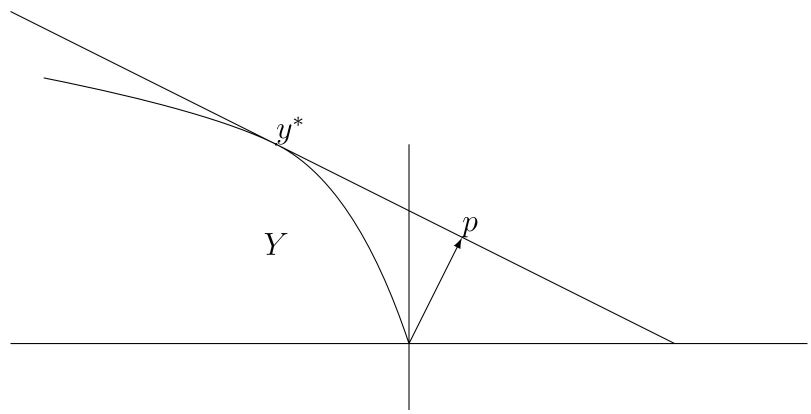

Figure 0.4.1. Maximizing p over Y.

References

[1] D. Gale. 1989. *Theory of linear economic models*. Chicago: University of Chicago Press. Reprint of the 1960 edition published by McGraw-Hill.