

Ec 181:
Convex Analysis and Economic Theory

KC Border

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Woe to the author who always wants to teach!
The secret of being a bore is to tell everything.

—Voltaire, *De la Nature de l'Homme* (1737)

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