Lecture 18: More topics in uncertainty

18.1 Investment

There are two assets, a safe asset that returns $(1+r_0)$ for each dollar invested and risky asset that returns $(1+\boldsymbol{r})$ for each dollar invested, where $\boldsymbol{r}$ is a nondegenerate random variable.

If his wealth is $\hat{w}$, an expected utility maximizing investor will choose the amount x to invest in the risky asset to maximize

$$\boldsymbol{E}\, u\Big((\hat{w}-x)(1+r_0)+x(1+\boldsymbol{r})\Big).$$

The difference $\boldsymbol{r}-r_0$ is the excess of $\boldsymbol{r}$ over the safe return, so for convenience, let us call it $\boldsymbol{q}$, i.e., $\boldsymbol{q}=\boldsymbol{r}-r_0$, and set $w=(1+r_0)\hat{w}$. Thus x is chosen to maximize

$$\boldsymbol{E}\, u(w+x\boldsymbol{q}),$$

which is a prettier problem.

There are some questions that are frequently glossed over in the literature. One is whether we want to restrict x to lie in the interval $[0,w]$. If so, we have to worry about boundary conditions. We also have to worry whether $w+x\boldsymbol{q}$ lies in the domain of the utility function with probability one. For instance, a utility function that is commonly studied is the logarithmic utility $u(w)=\ln w$ (where $u(0)=-\infty$ is allowed). If we make the limited liability assumption that $1+\boldsymbol{r}\geqslant 0$ a.s., and also restrict x to $[0,w]$, then we have no problems in that regard. On the other hand, we may actually want to allow borrowing ($x>w$) and/or short selling ($x<0$). In that case, we probably need to have the utility defined on the whole real line, which rules out the logarithmic utility, among others.

In what follows, I shall assume that utilities are defined on an interval D of the real line, are continuous strictly increasing functions on D that are twice continuously differentiable, with strictly positive derivatives everywhere on the interior of D, and that a solution exists and is interior to the domain.

The first order necessary condition for an interior maximum is

$$\boldsymbol{E}\, u'(w+x^*\boldsymbol{q})\boldsymbol{q}=0. \tag{$\star$}$$

Observe that $(\star)$ has a solution only if $\boldsymbol{q}<0$ with positive probability, which makes perfect economic sense. (Otherwise there would be an arbitrage opportunity: borrow at r_0 and invest at $\boldsymbol{r}$, earning a riskless profit.)

The second order necessary condition is

$$\boldsymbol{E}\, u''\Big(w+x^*\boldsymbol{q}\Big)\boldsymbol{q}^2\leqslant 0.$$

If u is concave, then $u'' \leqslant 0$, so the second order condition is automatically satisfied. I may also assume that the strong second order condition

$$\boldsymbol{E}\, u''\Big(w + x^*\boldsymbol{q}\Big)\boldsymbol{q}^2 < 0$$

holds at a particular solution. This is usually necessary to make the solution a differentiable function of the parameters.

18.1.1 A trivial lemma

18.1.1 Lemma *Let f be a nondecreasing real function on an interval I, let x belong to I, and let $\alpha > 0$. Then for any v for which $x + \alpha v \in I$, we have*

$$f(x + \alpha v)v \geqslant f(x)v.$$

This equality is reversed if $\alpha < 0$ or if f is nonincreasing. The inequality is strict provided $v \neq 0$ and f is not constant on the interval from x to $x + \alpha v$.

Proof: We prove the claim for $\alpha > 0$, the others are obvious from its proof. There are two interesting cases: $v > 0$ and $v < 0$. When $v > 0$, then the monotonicity of f implies $f(x + \alpha v) \geqslant f(x)$, so $f(x + \alpha v)v \geqslant f(x)v$. And when $v < 0$, then $f(x + \alpha v) \leqslant f(x)$, but multiplying by the negative quantity v reverses the inequality, so again $f(x + \alpha v)v \geqslant f(x)v$. ∎

18.1.2 Decreasing risk aversion

A natural comparative statics question is: What happens to x^* as a function of w?

18.1.2 Proposition *Assume u is C^2 and $u' > 0$, and define the Arrow–Pratt coefficient of risk aversion $r(w) = \dfrac{-u''(w)}{u'(w)}$. Fix w_0, and assume that x_0^* satisfies the strong second order condition. Then there is a neighborhood of w_0 on which x^* is a C^1 function of w.*

Moreover, if r is decreasing at w_0, then x^* is increasing at w_0 if x_0^* is positive and decreasing if x_0^* is negative. If, on the other hand, r is increasing at w_0, then $x^*(w)$ is decreasing when x_0^* is positive and increasing when x_0^* is negative.

Proof: Now x_0^* satisfies the first order condition

$$\boldsymbol{E}\, u'(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q} = 0.$$

By the strong second order condition, the Implicit Function Theorem implies that x^* is a C^1 function of w on an appropriate neighborhood of w_0. Thus differentiating the first order condition with respect to w gives

$$\boldsymbol{E}\, u''(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q}\left(1 + \boldsymbol{q}\frac{dx^*(w_0)}{dw}\right) = 0$$

or

$$\frac{dx^*(w_0)}{dw} = -\frac{\boldsymbol{E}\, u''(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q}}{\boldsymbol{E}\, u''(w + x_0^*\boldsymbol{q})\boldsymbol{q}^2}.$$

The strong second order condition implies that the denominator is negative so the sign of $\frac{dx^*(w_0)}{dw}$ is the sign of $\boldsymbol{E}\, u''(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q}$.

Now suppose $r(w)$ is decreasing at w_0. Consider first the case $x_0^* > 0$. By Lemma 18.1.1,

$$r(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q} \leqslant r(w_0)\boldsymbol{q}.$$

Therefore, recalling the definition of r and multiplying by the negative quantity $-u'(w_0 + x_0^*\boldsymbol{q})$, we have

$$u''(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q} \geqslant -r(w_0)u'(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q}.$$

Taking the expectation of both sides gives

$$\boldsymbol{E}\, u''(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q} \geqslant -r(w_0)\, \boldsymbol{E}\, u'(w_0 + x_0^*\boldsymbol{q})\boldsymbol{q} = 0$$

where the equality follows from the first order condition ($\star$). Thus

$$\operatorname{sign} \frac{dx^*(w_0)}{dw} = \operatorname{sign}\, Eu''(w + x_0^*\boldsymbol{q})\boldsymbol{q} \geqslant 0$$

when r is decreasing at w_0. Similarly, $\frac{dx^*(w_0)}{dw} \leqslant 0$ when r is increasing at w_0.

These conclusions are reversed if $x_0^* < 0$. ∎

18.1.3 What if u is more risk averse than v?

18.1.3 Proposition *Assume u is more risk averse than v. If v is risk averse or if the two preferences are "sufficiently close" (in a sense to be made precise in the proof), then*

$$0 \leqslant x_u^* \leqslant x_v^* \qquad \text{or} \qquad x_v^* \leqslant x_u^* \leqslant 0.$$

That is, the more risk averse utility adopts the more conservative portfolio.

Proof: We prove only the case $x_u^* \geqslant 0$. The other follows *mutatis mutandis*. Write $u = G \circ v$, where G is strictly increasing and concave. Then ($\star$) becomes

$$\boldsymbol{E}\, G'\Big(v(w + x_u^*\boldsymbol{q})\Big)v'(w + x_u^*\boldsymbol{q})\boldsymbol{q} = 0.$$

Since G is concave, G' is nonincreasing, and thus so is $G' \circ v$. By Lemma 18.1.1,

$$G'\Big(v(w + x_u^*\boldsymbol{q})\Big)\boldsymbol{q} \leqslant G'\Big(v(w)\Big)\boldsymbol{q}.$$

Since $v' > 0$, we have

$$G'\Big(v(w + x_u^*\boldsymbol{q})\Big)v'(w + x_u^*\boldsymbol{q})\boldsymbol{q} \leqslant G'\Big(v(w)\Big)v'(w + x_u^*\boldsymbol{q})\boldsymbol{q},$$

and taking expectations yields

$$\underbrace{\boldsymbol{E}\, G'(v(w + x_u^*\boldsymbol{q}))v'(w + x_u^*\boldsymbol{q})\boldsymbol{q}}_{=0 \text{ by } (\star)} \leqslant G'(v(w))\, \boldsymbol{E}\, v'(w + x_u^*\boldsymbol{q})\boldsymbol{q}.$$

That is,

$$\boldsymbol{E}\, v'(w + x_u^*\boldsymbol{q})\boldsymbol{q} \geqslant 0.$$

But the first order condition for x_v^* is

$$\boldsymbol{E}\, v'(w + x_v^*\boldsymbol{q})\boldsymbol{q} = 0.$$

Now set $h(x) = \boldsymbol{E}\, v'(w + x\boldsymbol{q})\boldsymbol{q}$. Then $h(x_u^*) \geqslant 0 = h(x_v^*)$. But $h'(x_v^*) = \boldsymbol{E}\, v''(w + x_v^*\boldsymbol{q})\boldsymbol{q}^2 \leqslant 0$ by the second order condition for x_v^*. If u and v are close enough so that $h'(x) \leqslant 0$ on the interval between x_v^* and x_u^*, then $x_u^* \leqslant x_v^*$. (If v is concave, then $h' \leqslant 0$ and no closeness assumption is needed.) ∎

18.2 Deductibles vs. Coinsurance

You are subject to two kinds of risk. With probability $p_1 > 0$ you lose an amount x_1, and with probability $p_2 > 0$ you lose x_2. Assume $x_2 > x_1 > 0$ and $1 - p_1 - p_2 > 0$.

An insurance company offers two kinds of policies. The deductible policy reimburses you for all but d of your loss. The coinsurance policy reimburses you a fraction $1 - \rho$ of your loss. Suppose $0 < d < x_1 < x_2$ and that both policies have the same premium $\pi > 0$, and that both policies have the same expected value.

Suppose you are a risk averse expected utility maximizer and face no other risks. Which policy do you prefer?

Answer: Let w denote your initial wealth. There are three states of the world $\{0, 1, 2\}$. The random variables representing your wealth under the two policies are:

state	deductible X_d	coinsurance X_c	difference Z
0	$w - \pi$	$w - \pi$	0
1	$w - \pi - d$	$w - \pi - \rho x_1$	$d - \rho x_1$
2	$w - \pi - d$	$w - \pi - \rho x_2$	$d - \rho x_2$

That is,

$$X_c = X_d + Z.$$

Now observe that $Z = 0$ in the event $X_d = w - \pi$, and conditional on the event $X_d = w - \pi - d$ the expectation of Z is $\big((p_1 + p_2)d - \rho(p_1x_1 + p_2x_2)\big)/(p_1 + p_2)$. But both policies have the same expected value, $(p_1 + p_2)d = \rho(p_1x_1 + p_2x_2)$. Therefore

$$E(Z|X_d) = 0.$$

Then X_c is riskier than X_d, so a risk averse expected utility prefers X_d to X_c.

A less elegant but more elementary argument runs like this: Let U denote your utility and w denote your wealth. The expected utilities of the policies are:

$$\begin{aligned} EU_{\text{deductible}} &= (1-p_1-p_2)U(w-\pi) + (p_1+p_2)U(w-\pi-d) \\ EU_{\text{coinsurance}} &= (1-p_1-p_2)U(w-\pi) \\ &\quad + p_1 U(w-\pi-\rho x_1) + p_2 U(w-\pi-\rho x_2) \end{aligned}$$

Since the policies have the same expected value,

$$p_1(x_1-d) + p_2(x_2-d) = (1-\rho)(p_1x_1+p_2x_2).$$

Rearranging,

$$-d = -\rho\left(\frac{p_1}{p_1+p_2}x_1 + \frac{p_2}{p_1+p_2}x_2\right),$$

so

$$\begin{aligned} w-\pi-d &= w-\pi-\rho\left(\frac{p_1}{p_1+p_2}x_1 + \frac{p_2}{p_1+p_2}x_2\right) \\ &= \frac{p_1}{p_1+p_2}(w-\pi-\rho x_1) + \frac{p_2}{p_1+p_2}(w-\pi-\rho x_2). \end{aligned}$$

Since U is concave,

$$\begin{aligned} U(w-\pi-d) &= U\left(\frac{p_1}{p_1+p_2}(w-\pi-\rho x_1) + \frac{p_2}{p_1+p_2}(w-\pi-\rho x_2)\right) \\ &\geqslant \frac{p_1}{p_1+p_2}U(w-\pi-\rho x_1) + \frac{p_2}{p_1+p_2}U(w-\pi-\rho x_2) \end{aligned}$$

Multiply both sides by p_1+p_2 and add $(1-p_1-p_2)U(w-\pi)$ to conclude that

$$EU_{\text{deductible}} \geqslant EU_{\text{coinsurance}}.$$

18.3 Alternative models

Multiple probability (MP) models typically rank random variables according to a function of the form

$$V(X) = \min_{P\in\mathcal{P}} \int_S u\big(X(s)\big)\, dP(s),$$

where $\mathcal{P}$ is a set of probabilities. If $\mathcal{P}$ includes all the degenerate probabilities ($\delta_s(\{s\}) = 1$), then this reduces to the **maximin** criterion, which ranks according to $\min_s X(s)$.

Another model is the **Choquet expected utility** (CEU) model, which uses a function of the form

$$V(X) = \int \nu[X > t]\,dt,$$

where ν is a *Choquet capacity* (a function on events satisfying $E \subset F \implies \nu(E) \leqslant \nu(F)$, but is not necessarily additive). By Proposition 16.12.2, if ν is a probability, this agrees with the usual expected utility. It is designed to explain the *Ellsberg paradox* and capture *ambiguity aversion*.

18.4 State-preference diagrams

In this section we shall consider only strongly risk averse EU decision makers with Bernoulli utility u satisfying $u' > 0$ and $u'' < 0$.

A two-valued random variable can be represented as a point $X = (x_a, x_b)$ in $\boldsymbol{R}^2$ (the value in event a is x_a and in event b is x_b). The diagonal $\{(x,x) : x \in \boldsymbol{R}\}$ is called the **certainty line**, the value of X is the same in either event.

An **indifference curve** is a set of random variables with the same expected utility. That is, the set of pairs (x, y) such that

$$p_a u(x) + p_b u(y) = \text{constant},$$

where p_a is the probability of event a, etc. For each x, let $\hat{y}(x)$ satisfy

$$p_a u(x) + p_b u\big(\hat{y}(x)\big) = \text{constant}.$$

Since the left-hand side is independent of x, its derivative with respect to x must be zero. That is,

$$p_a u'(x) + p_b u'\big(\hat{y}(x)\big)\hat{y}'(x) = 0, \tag{1}$$

so the slope of the indifference curve at the point $\big(x, \hat{y}(x)\big)$ is

$$\hat{y}'(x) = -\frac{p_a u'(x)}{p_b u'\big(\hat{y}(x)\big)} < 0.$$

Along the certainty line we have $\hat{y}(x) = x$, so the slope there is just $-p_a/p_b$. That is,

> for an EU decision maker, all the indifference curves have the same slope where they cross the certainty line, namely $-p_a/p_b$.

Now equation (1) holds for all x, so differentiating with respect to x gives

$$p_a u''(x) + p_b u''\big(\hat{y}(x)\big)\big(\hat{y}'(x)\big)^2 + p_b u'\big(\hat{y}(x)\big)\hat{y}''(x) = 0,$$

so

$$\hat{y}''(x) = -\frac{p_a u''(x) + p_b u''\big(\hat{y}(x)\big)\big(\hat{y}'(x)\big)^2}{p_b u'\big(\hat{y}(x)\big)} > 0$$

since $u' > 0$ and $u'' < 0$. That is, the indifference curve has a convex shape.

18.4.1 Bets on a

A **bet on a** is a random variable $X = (x_a, -x_b)$ with $x_a, x_b > 0$. A bet is **fair** if its expectation is zero, which entails

$$p_a x_a - p_b x_b = 0, \qquad \text{or} \qquad \frac{p_a}{p_b} = \frac{x_b}{x_a}.$$

A bet is **favorable** if

$$p_a x_a - p_b x_b > 0, \qquad \text{or} \qquad \frac{p_a}{p_b} > \frac{x_b}{x_a}.$$

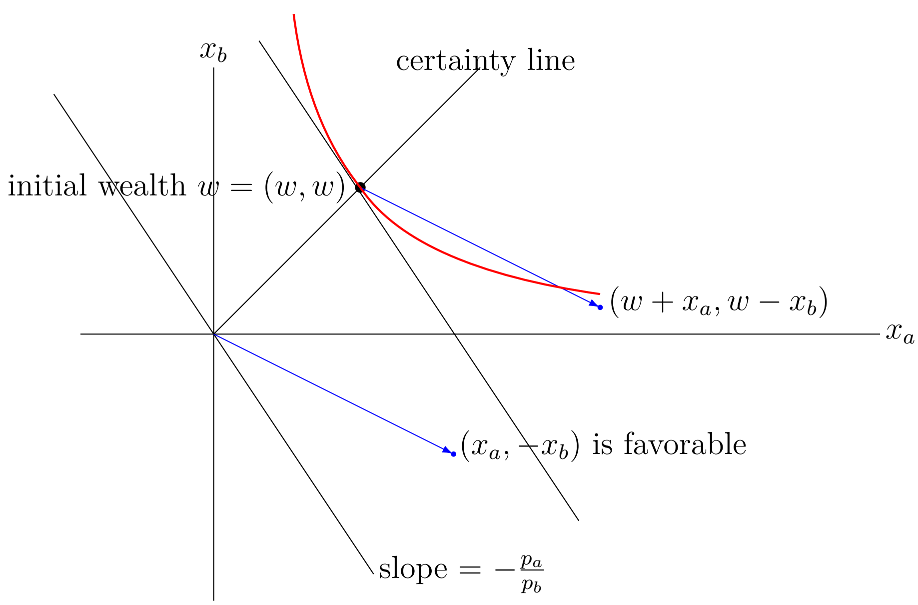

Figure 18.1.

Suppose a risk averse EU dm with wealth $W = (w, w)$ on the certainty line) is offered the favorable bet X. The line connecting W and $W + X$ has slope $-x_b/x_a$, while lines of equal expectation have slope $-p_a/p_b$. See Figure 18.1.

If their indifference curve is as drawn, the dm will not want to take the bet, since it would put them on a lower indifference curve. But since their indifference curve has slope $-p_a/p_b$ at W, the line segment joining W and $W + X$ crosses higher indifference curves, so for small enough $\lambda > 0$, the point $W + \lambda X$ is preferred to W. So the dm would prefer to be able to take the bet λX. This demonstrates the following proposition.

18.4.1 Proposition *A risk averse EU dm with a smooth Bernoulli utility will prefer to take a small part of any favorable bet.*

We can also derive the same result as follows. Let α be the fraction of the bet X, and choose it to maximize the expected utility

$$p_a u(w + \alpha x_a) + p_b u(w - \alpha x_b).$$

The first order condition is

$$p_a u'(w + \alpha^* x_a)x_a - p_b u'(w - \alpha^* x_b)x_b = 0,$$

so

$$\frac{p_a x_a}{p_b x_b} = \frac{u'(w - \alpha^* x_b)}{u'(w + \alpha^* x_a)}.$$

If the bet is favorable, then $p_a x_a > p_b x_b$, so the above implies $\frac{u'(w-\alpha^* x_b)}{u'(w+\alpha^* x_a)} > 1$, and since u' is decreasing for a risk averter, we have $\alpha^* > 0$.

References

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