Lecture 17: Risk and Risk Aversion

17.1 Risk aversion in the EU model

Risk aversion is the (weak) preference for $\boldsymbol{E}\, X$ for sure to X for all nondegenerate random variables X. That is,

$$u(\boldsymbol{E}\, X) \geqslant \boldsymbol{E}\, u(X).$$

In particular, if an EU dm with utility u is risk averse, and X assumes the values x and y with probabilities $1-p$ and p respectively, then

$$u\Big((1-p)x + py\Big) \geqslant u\Big((1-p)x + py\Big).$$

In other words, u is **concave**. Conversely if u is concave, then the dm is risk averse, which is a mathematical result known as *Jensen's inequality*.

In practice, it is easiest to identify concave functions by their derivatives. A differentiable utility u is concave if and only $u'(x)$ is a monotone decreasing function of x. A twice-differentiable utility u is concave if and only $u''(x) \leqslant 0$ for all x. Note that linear functions are concave. A dm with a linear utility is **risk neutral** and ranks random variables according to their expectation.

17.2 Stochastic dominance

The rv X **stochastically dominates** Y if

$$\boldsymbol{E}\, u(X) \geqslant \boldsymbol{E}\, u(Y) \text{ for every monotone nondecreasing function } u.$$

17.2.1 Theorem *X stochastically dominates Y if and only if*

$$F_X(t) \leqslant F_Y(t) \text{ for all } t.$$

17.3 Riskiness

The rv X is **riskier** than Y if

$$\boldsymbol{E}\, u(X) \leqslant \boldsymbol{E}\, u(Y) \text{ for every concave function } u.$$

Suppose the supports of F_X and F_Y satisfy $F(a) = F_Y(a) = 0$ and $F(b) = F_Y(b) = 1$.

17.3.1 Theorem *The following are equivalent.*

$$\forall s \in [a,b] \quad \int_a^s F_X(t)\,dt \geqslant \int_a^s F_Y(t)\,dt \quad \& \int_a^b F_X(t)\,dt = \int_a^b F_Y(t)\,dt \tag{1}$$

$$\boldsymbol{E}\,u(X) \leqslant \boldsymbol{E}\,u(Y) \text{ for every concave function } u. \tag{2}$$

$$X = Y + Z \qquad \text{where } \boldsymbol{E}(Z|Y) = 0. \tag{3}$$

Proof that (2) implies (1): Let $s \in [a,b]$. Integrating by parts,

$$\begin{aligned}\int_a^s F_X(t)\,dt &= tF_X(t)\Big|_a^s - \int_a^s t\,dF_X(t).\\ &= sF_X(s) - \int_a^s t\,dF_X(t)\\ &= \int_a^s (s-t)\,dF_X(t)\\ &= \int_a^b (s-t)^+\,dF_X(t).\end{aligned}$$

Similarly

$$\int_a^s F_Y(t)\,dt = \int_a^b (s-t)^+\,dF_Y(t).$$

Since $(s-t)^+$ is a convex function of t, (2) implies

$$\int_a^s F_X(t)\,dt = \int_a^b (s-t)^+\,dF_X(t) \geqslant \int_a^b (s-t)^+\,dF_Y(t) = \int_a^s F_Y(t)\,dt.$$

When $s = b$, this becomes $\int_a^b F_X(t)\,dt = \int_a^b (b-t)\,dF_X(t)$. Now $b-t$ is both convex and concave in t, so we must have $\int_a^b F_X(t)\,dt = \int_a^b F_Y(t)\,dt$. ∎

17.4 Comparative risk aversion

A risk averse dm will pay to eliminate risk. We will say that one dm is **more risk averse** than another if his willingness to pay is always higher. Specifically, define **risk premium** $\pi_u(w,Z)$ by the equation

$$u\Big(w + \boldsymbol{E}\,Z - \pi_u(w,Z)\Big) = \boldsymbol{E}\,u(w+Z). \tag{$\star$}$$

It is the most that an EU decision maker with von Bernoulli utility function u would be willing to pay to completely insure against the risk Z to his initial wealth w.

When u is twice differentiable, the **(Arrow–Pratt) coefficient of risk aversion** r_u is defined by

$$r_u(w) = -\frac{u''(w)}{u'(w)}.$$

Note that this coefficient is invariant under positive affine transformations of u, so it really is a property of the preferences.

17.4.1 Theorem *Let u and v be continuous strictly increasing functions that are twice differentiable with strictly positive derivatives. Then the following statements are equivalent.*

1. *For all w and all random variables Z that satisfy $\boldsymbol{E}\, Z = 0$,*

$$\pi_u(w, Z) \geqslant \pi_v(w, Z).$$

2. *There exists a concave strictly increasing function g defined on the range of v satisfying*

$$u = g \circ v.$$

3. *For all w,*

$$-\frac{u''(w)}{u'(w)} \geqslant -\frac{v''(w)}{v'(w)}.$$

For a pedantic proof, see my on-line notes [2].

17.4.1 Intepretation of the Arrow–Pratt-deFinetti coefficient

For each ε small enough let Z_ε be a random variable that takes on each of the values ε and $-\varepsilon$ with probability $\frac{1}{2}$. Then $\boldsymbol{E}\, Z_\varepsilon = 0$ and Z_ε is admissible for u at w. To simplify notation, define the real function p on A by $p(\varepsilon) = \pi_u(w, Z_\varepsilon)$. Note that $p(0) = 0$, $p(\varepsilon) = p(-\varepsilon)$, and by definition,

$$u\Big(w - p(\varepsilon)\Big) = \tfrac{1}{2}u(w + \varepsilon) + \tfrac{1}{2}u(w - \varepsilon). \tag{4}$$

Note that (4) implies that the function p is twice differentiable on A.[1] But since the risk premium is unique this function must be p.

Since (4) holds for all small ε, we may differentiate both sides to get

$$-u'\Big(w - p(\varepsilon)\Big)p'(\varepsilon) = \tfrac{1}{2}u'(w + \varepsilon) - \tfrac{1}{2}u'(w - \varepsilon).$$

In particular, $p'(0) = 0$. Differentiating a second time yields

$$u''\Big(w - p(\varepsilon)\Big)\Big(p'(\varepsilon)\Big)^2 - p''(\varepsilon)u'\Big(w - p(\varepsilon)\Big) = \tfrac{1}{2}u''(w + \varepsilon) + \tfrac{1}{2}u''(w - \varepsilon).$$

[1] To see this, define the function $f\colon A \times (D - w) \to \boldsymbol{R}$ by

$$f(\varepsilon, \eta) = u\big(w - \eta\big) - \tfrac{1}{2}u(w + \varepsilon) - \tfrac{1}{2}u(w - \varepsilon) \tag{5}$$

and note that f is twice differentiable, $f(0,0) = 0$, and $\dfrac{\partial f(0,0)}{\partial \eta} = -u'(w) < 0$. Therefore by the Implicit Function Theorem (see, e.g., [4, Theorem 2, p. 235]) there is a unique twice differentiable function defined on a neighborhood of zero giving η as a function of ε to satisfy equation (5).

In particular, using $p(0) = p'(0) = 0$, we have

$$p''(0) = -\frac{u''(w)}{u'(w)} = r_u(w).$$

We can apply Taylor's Theorem [3, p. 290] to write

$$\begin{aligned} p(\varepsilon) &= p(0) + \varepsilon p'(0) + \frac{\varepsilon^2}{2}\Big(p''(0) + r(\varepsilon)\Big) \\ &= \frac{\varepsilon^2}{2}\Big(p''(0) + r(\varepsilon)\Big), \end{aligned} \tag{6}$$

where $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$.[2]

Now the variance of Z_ε is ε^2. So $\frac{p(\varepsilon)}{\varepsilon^2}$ is the fraction of the variance that someone with utility u would be willing to pay to insure against Z_ε. The limit of this fraction as $\varepsilon \to 0$ is then $\frac{1}{2} r_u(w)$. In fact, this generalizes to more general admissible small random variables with variance $\varepsilon > 0$.

17.4.2 Sketch of Proof of Theorem 17.4.1

(1) $\implies$ (3): Using equation (6) and $p_u \geqslant p_v$, we conclude

$$p''_u(0) + R_u(\varepsilon) \geqslant p''_v(0) + R_v(\varepsilon)$$

for all $\varepsilon > 0$ in A. Letting $\varepsilon \downarrow 0$, we conclude that

$$p''_u(0) \geqslant p''_v(0).$$

That is,

$$-\frac{u''(w)}{u'(w)} \geqslant -\frac{v''(w)}{v'(w)}.$$

(3) $\implies$ (2): Since u and v are both strictly increasing, the function

$$g = u \circ v^{-1}$$

is strictly increasing and satisfies $u = g \circ v$. What we need to show is that g is strictly concave, which we shall accomplish by showing that $g'' < 0$. Now $u(w) = g\big(v(w)\big)$ for all w, we may differentiate both sides to conclude

$$u'(w) = g'\Big(v(w)\Big)v'(w) \tag{7}$$

and

$$u''(w) = g''\Big(v(w)\Big)\Big(v'(w)\Big)^2 + g'\Big(v(w)\Big)v''(w). \tag{8}$$

[2] The form of Taylor's Theorem given by Hardy [3] requires only twice differentiability at 0, not twice continuous differentiability on a neighborhood of 0.

Dividing (8) by (7) yields

$$\frac{u''(w)}{u'(w)} = \frac{g''\big(v(w)\big)}{g'\big(v(w)\big)} v'(w) + \frac{v''(w)}{v'(w)}.$$

Thus $r_u \geqslant r_v$ implies $\dfrac{g''\big(v(w)\big)}{g'\big(v(w)\big)} v'(w) \leqslant 0$, but $u' > 0$ and $v' > 0$ imply $g'\big(v(w)\big) = \dfrac{u'\big(v(w)\big)}{v'(w)} > 0$, so $g''\big(v(w)\big) \leqslant 0$. Thus g is concave.

(2) $\implies$ (1): This is clearly a job for Jensen's inequality. Let $\boldsymbol{E}\, Z = 0$. Observe that

$$\begin{aligned} u\Big(w - \pi_u(w,Z)\Big) &= \boldsymbol{E}\, u(w+Z) \\ &= \boldsymbol{E}\, g\Big(v(w+Z)\Big) \\ &\leqslant g\Big(\boldsymbol{E}\, v(w+Z)\Big) \\ &= g\bigg(v\Big(w - \pi_v(w,Z)\Big)\bigg) \\ &= u\Big(w - \pi_v(w,Z)\Big), \end{aligned}$$

where the inequality follows from Jensen's inequality and the concavity of g, and the other equalities are either from the definition of the risk premium or the assumption that statement (2) holds.

Since u is strictly increasing it follows that $\pi_u(w,Z) \geqslant \pi_v(w,Z)$.

17.5 Recovering the utility from the risk aversion

We can recover the utility function from the coefficient of risk aversion.

17.5.1 Theorem *Let $r\colon D \to \boldsymbol{R}$ be a continuous function on an open interval D. Then there is a C^2 utility $u\colon D \to \boldsymbol{R}$ satisfying $u'(w) > 0$ for all $w \in D$ and*

$$-\frac{u''(w)}{u'(w)} = r(w) \quad \text{for all } w \in D.$$

The function u is unique up to positive affine transformation and is given by

$$u(w) = \alpha \int_a^w e^{-R(x)}\, dx + \beta$$

where $a \in D$, $\alpha > 0$,

$$R(x) = \int_a^x r(t)\, dt,$$

and β is an arbitrary real number.

Proof: Observe that we are seeking a solution to the linear differential equation

$$u''(x) - r(x)u'(x) = 0.$$

Letting $f(x)$ denote $u'(x)$, this becomes the first order equation

$$f'(x) + r(x)f(x) = 0.$$

It is well known, see, e.g, Apostol [1, Theorem 8.3, p. 310], that for each $a \in D$ and $\alpha \in \boldsymbol{R}$, this equation has a unique solution satisfying $f(a) = \alpha$. It is given by

$$f(x) = \alpha e^{-R(x)}, \qquad \text{where } R(x) = \int_a^x r(t)\,dt.$$

Any such solution is continuous, so by the same result (now with $P = 0$ and $Q = f$), the differential equation

$$u'(x) = f(x)$$

has a solution u given by

$$\begin{aligned} u(x) &= \int_a^x f(t)\,dt + \beta \\ &= \alpha \int_a^x e^{-R(t)}\,dt + \beta. \end{aligned}$$

To guarantee that $u'(x) > 0$, we need only take $\alpha > 0$, as $e^{-R(x)} > 0$. Thus within the class of strictly increasing functions, given a, the utility u is unique up to the constants $\alpha > 0$ and β. ∎

17.5.2 Example (Constant absolute risk aversion (cara)) Suppose the risk aversion function is constant:

$$r(x) = c$$

for all x. Then assuming $a = 0$ is in the domain, we have $R(x) = cx$, so

$$u(x) = \alpha \int_0^x e^{-ct}\,dt + \beta.$$

If $c = 0$, then $\int_0^x e^{-0t}\,dt = \int_0^x 1\,dt = x$

$$u(x) = \alpha x + \beta.$$

If $c \neq 0$, then $R(x) = cx$, and the primitive of $e^{-cx} = -\frac{1}{c}e^{-cx}$, so $\int_0^x e^{-ct}\,dt = -\frac{1}{c}e^{-cx} - (-\frac{1}{c}e^{-c0}) = -\frac{1}{c}\Big(e^{-cx} - 1\Big)$, so our formula gives

$$u(x) = -\frac{\alpha}{c}\Big(e^{-cx} - 1\Big) + \beta.$$

We can choose particularly aesthetic values of α and β. For $c = 0$, choose $\alpha = 1$ and $\beta = 0$, to get

$$u(x) = x \quad (c = 0).$$

For $c > 0$, set $\beta = 0$ and $\alpha = c > 0$ to get

$$u(x) = 1 - e^{-cx} \quad (c > 0).$$

For $c < 0$, set $\beta = 0$ and $\alpha = -c > 0$, to get

$$u(x) = e^{-cx} - 1 \quad (c < 0).$$

□

References

[1] T. M. Apostol. 1967. *Calculus*, 2d. ed., volume 1. Waltham, Massachusetts: Blaisdell.

[2] K. C. Border. 1996. Pedantic notes on the risk premium and risk aversion. http://www.its.caltech.edu/~kcborder/Notes/RiskAversion.pdf

[3] G. H. Hardy. 1952. *A course of pure mathematics*, 10th. ed. Cambridge: Cambridge University Press.

[4] J. E. Marsden. 1974. *Elementary classical analysis*. San Francisco: W. H. Freeman and Company.