Lecture 14: Topics in Demand Theory

14.1 Money metric indirect utility

Define

$$\mu(p^*; p, m) = e\Big(p^*, \mathrm{v}(p, m)\Big), \tag{$\star$}$$

where p^* is an arbitrary price vector. Since $e(p, v)$ is strictly increasing in v, this is an **indirect utility** or **welfare measure**. That is,

$$\mu(p^*; p, m) \geqslant \mu(p^*; p', m') \quad \text{if and only if} \quad \mathrm{v}(p, m) \geqslant \mathrm{v}(p', m'),$$

but the units of μ are in dollars (or euros, or whatever). The money values depend on the choice of p^*. This function is variously called a **money metric (indirect) utility** or an **income-compensation function**.

14.2 Compensating and equivalent variation

Consider a budget change from (p^0, m^0) to (p^1, m^1) (say caused by the repeal of the corn laws). In 1942 John Hicks [4] defined the **compensating variation** (CV) to be the amount that you would have to deduct from a consumer's income to leave him/her exactly as well off as before the budget change.[1] This amount to comparing the budgets with the money metric indirect utility using the new prices. It is positive if the new budget makes the consumer better off (remember it is a deduction).

Hicks also defined the **equivalent variation** (EV) to be the increase in welfare due to the change in budget, measured using the original prices. It may be negative if the new budget (p^1, m^1) is worse than the original (p^0, m^0).

Thus

$$\mathrm{EV}(p^0, m^0; p^1, m^1) = \mu(p^0; p^1, m^1) - \underbrace{\mu(p^0; p^0, m^0)}_{m^0} \tag{1}$$

$$\mathrm{CV}(p^0, m^0; p^1, m^1) = \underbrace{\mu(p^1; p^1, m^1)}_{m^1} - \mu(p^1; p^0, m^0) \tag{2}$$

Or letting

$$v_0 = \mathrm{v}(p^0, m^0) \qquad v_1 = \mathrm{v}(p^1, m^1)$$

[1] Actually Hicks only considered price changes of the form (p^0, m) to (p^1, m) holding income fixed. The obvious generalization here is discussed by Chipman and Moore [1] in 1980.

we have the equivalent

$$\begin{aligned}\mathrm{EV}(p^0, m^0; p^1, m^1) &= e(p^0, v_1) - e(p^0, v_0) \\ &= e(p^0, v_1) - m^0 \qquad (1')\end{aligned}$$

$$\begin{aligned}\mathrm{CV}(p^0, m^0; p^1, m^1) &= e(p^1, v_1) - e(p^1, v_0) \\ &= m^1 - e(p^1, v_0). \qquad (2')\end{aligned}$$

14.3 A single price change

Now consider only a decrease in the price of good i. That is,

$$p_i^1 < p_i^0,$$
$$p_j^1 = p_j^0 = \bar{p}_j \text{ for } j \neq i$$

and

$$m^0 = m^1 = m.$$

Since the price of good i decreases, the new budget set includes the old one, so welfare will increase.

Then

$$\mathrm{EV}(p^0, p^1) = \mu(p^0; p^1, m^1) - \underbrace{\mu(p^0; p^0, m^0)}_{m}$$

$$\mathrm{CV}(p^0, p^1) = \underbrace{\mu(p^1; p^1, m^1)}_{m} - \mu(p^1; p^0, m^0)$$

We now use the following **trick** (which applies whenever $m^0 = m^1 = m$):

$$\mu(p^0; p^0, m^0) = m = \mu(p^1; p^1, m^1). \qquad (3)$$

This enables us to rewrite the values as

$$\mathrm{EV} = \mu(p^0; p^1, m^1) - \underbrace{\mu(p^1; p^1, m^1)}_{m}$$

$$\mathrm{CV} = \underbrace{\mu(p^0; p^0, m^0)}_{m} - \mu(p^1; p^0, m^0)$$

Or in terms of the expenditure function, we have:

$$\begin{aligned}\mathrm{EV} &= \mu(p^0; p^1, m^1) - \mu(p^1; p^1, m^1) \\ &= e(p^0, v_1) - e(p^1, v_1) \qquad (1'')\end{aligned}$$

$$\begin{aligned}\mathrm{CV} &= \mu(p^0; p^0, m^0) - \mu(p^1; p^0, m^0) \\ &= e(p^0, v_0) - e(p^1, v_0) \qquad (2'')\end{aligned}$$

Now we use the fact that $\partial e/\partial p_i = \tilde{x}_i$ to get

$$\text{EV} = e(p^0, v_1) - e(p^1, v_1) = \int_{p_i^1}^{p_i^0} \tilde{x}_i(p, \bar{p}_{-i}, v_1)\,dp$$

$$\text{CV} = e(p^0, v_0) - e(p^1, v_0) = \int_{p_i^1}^{p_i^0} \tilde{x}_i(p, \bar{p}_{-i}, v_0)\,dp,$$

where the notation $p, \bar{p}_{-i}$ refers to the price vector $(\bar{p}_1, \dots, \bar{p}_{i-1}, p, \bar{p}_{i+1}, \dots, p_n)$. Also since $p_i^0 > p_i^1$, the integrals above are positive if $\tilde{x}_i$ is positive.

Now by the equivalence of expenditure minimization and utility maximization we know that

$$x^*(p^1, m) = \tilde{x}(p^1, v_1) \quad \text{and} \quad x^*(p^0, m) = \tilde{x}(p^0, v_0)$$

From the Slutsky equation

$$\frac{\partial \tilde{x}_i}{\partial p_j} = \frac{\partial x_i^*}{\partial p_j} - x_j^* \frac{\partial x_i^*}{\partial m}.$$

Assume now that good i is not inferior, that is, assume

$$\frac{\partial x_i^*}{\partial m} > 0.$$

Then the Slutsky equation tells us that

$$0 > \frac{\partial \tilde{x}_i(p^1, m)}{\partial p_i} > \frac{\partial x_i^*(p^1, v_1)}{\partial p_i} \quad \text{and} \quad 0 > \frac{\partial \tilde{x}_i(p^0, m)}{\partial p_i} > \frac{\partial x_i^*(p^0, v_0)}{\partial p_i}$$

That is, the ordinary demand x_i^* is steeper than the Hicksian demand $\tilde{x}_i$ and cuts across as in Figure 14.1. As you can see, for a price decrease,

$$\text{EV} > \text{CS} > \text{CV}.$$

The inequalities are reversed for a price increase. Also note that if $\frac{\partial x_i^*}{\partial m} = 0$, then the three demand curves coincide.

14.4 A caveat

While the income-compensation $\mu(p^*; p, m)$ defined by ($\star$) is a valid welfare measure, it must be used carefully and consistently.

For instance, is the following true?

$$¿¿\ \text{CV}(p^0, m^0; p^1, m^1) + \text{CV}(p^1, m^1; p^2, m^2) = \text{CV}(p^0, m^0; p^2, m^2)\ ??$$

The answer is **no**, except for certain special classes of utility functions. See Chipman and Moore [1] and the references therein.

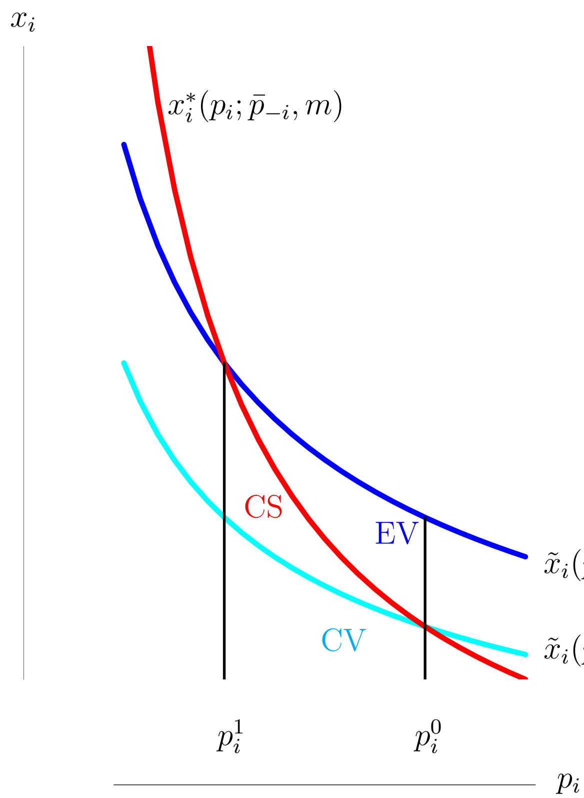

Figure 14.1. Illustration of a single price change. (Graphs are for a Cobb–Douglas utility.)

N.B. The horizontal axis is the price axis and the vertical axis is quantity axis.

The equivalent variation is the area under the Hicksian demand curve for utility level υ_0.
The compensating variation is the area under the Hicksian demand curve for utility level υ_1.
The consumer's surplus is the area under the ordinary demand curve.

14.5 "Deadweight loss"

Consider a simple problem where the good 1 is taxed an *ad rem* tax t, but income remains unchanged. The original price vector is p^0 and the new one is $p^1 = p^0 + te^1$. The tax revenue T collected is

$$T = tx_1^*(p^1, m).$$

Clearly the consumer is worse off under the price vector p^1. Let's use a money metric indirect utility to find the dollar value of the loss and compare it to the tax revenue. Let's choose $p^* = p^0$, which makes the welfare loss the negative of the equivalent variation. The dollar value of the welfare loss is

$$L = \mu(p^0; p^0, m) - \mu(p^0; p^1, m).$$

Let's write this in terms of the expenditure function as

$$L = e\Big(p^0, \mathrm{v}(p^0, m)\Big) - e\Big(p^0, \mathrm{v}(p^1, m)\Big). \tag{4}$$

Let $\upsilon_1 = \mathrm{v}(p^1, m)$, be the consumer's utility under the tax. Then

$$\mu(p^0; p^1, m) = e(p^0, \upsilon_1).$$

Now the equivalence of utility maximization and expenditure minimization implies $e\Big(p^0, \mathrm{v}(p^0, m) = m$ (cf. Section 13.1). Now we use our trick (3) that $m = \mu(p^1; p^1, m^1) = e(p^1, \upsilon_1)$ to rewrite (4)

$$L = e(p^1, \upsilon_1) - e(p^0, \upsilon_1), \tag{4$'$}$$

which agrees with (2″) as the expression for the negative of the equivalent variation. Then

$$\begin{aligned} L &= e(p_1^0 + t, p_2^0, \ldots, p_n^0, \upsilon_1) - e(p_1^0, p_2^0, \ldots, p_n^0, \upsilon_1) \\ &= \int_{p_1^0}^{p_1^0+t} \frac{\partial e(p, p_2^0, \ldots, p_n^0, \upsilon_1)}{\partial p_1} \, dp \\ &= \int_{p_1^0}^{p_1^0+t} \tilde{x}_1(p, p_2^0, \ldots, p_n^0, \upsilon_1) \, dp \end{aligned}$$

but Hicksian compensated demands are downward sloping, so

$$\begin{aligned} &> \int_{p_1^0}^{p_1^0+t} \tilde{x}_1(p_1^0 + t, p_2^0, \ldots, p_n^0, \upsilon_1) \, dp \\ &= t\tilde{x}_1(p_1^0 + t, p_2^0, \ldots, p_n^0, \upsilon_1) \\ &= tx_1^*(p^1, m) \\ &= T. \end{aligned}$$

The welfare loss L is greater than the tax revenue T.

The difference $L - T$ is called **deadweight loss**[2] from *ad rem* taxation.

Suppose instead there was a lump sum tax T which did not change the prices. Then the welfare loss is

$$e\Big(p^0, \mathrm{v}(p^0, m)\Big) - e\Big(p^0, \mathrm{v}(p^0, m - T)\Big) = T.$$

> The amazing thing is not so much that the *ad rem* tax is inferior to the lump-sum tax, but that *some taxes are worse than others* at all, even when they collect the same amount of revenue! This would not be apparent without our theoretical apparatus.

14.6 Revealed preference and lump-sum taxation

Recall that x is **revealed preferred** to y if there is some budget containing both x and y and x is chosen. If the choice function is generated by utility maximization, then if x is revealed preferred to y, we must have $u(x) \geqslant u(y)$.

So consider an *ad rem* tax t on good 1 versus a lump-sum tax, as above. Assume both taxes raise the same revenue T. The *ad rem* tax leads to the budget (p^1, m), and the lump-sum to the budget (p^0, m), where

$$p^1_1 = p^0_1 + t \quad \text{and} \quad p^1_j = p^0_j,\ j = 2, \dots, n.$$

Let x^1 be demanded under the *ad rem* tax and x^0 be demanded under the lump-sum tax. Then x^0 is revealed preferred to x^1:

$$m \geqslant p^1 \cdot x^1 = p^0 \cdot x^1 + t x^1_1 = p^0 \cdot x^1 + T,$$

so

$$p^0 \cdot x^1 \leqslant m - T,$$

which says that x^1 is in the budget $(p^0, m - T)$, from which x^0 is chosen. Thus

$$u(x^0) \geqslant u(x^1),$$

so the lump-sum tax is at least as good as the *ad rem* tax.

This argument is a lot simpler than the argument above, but we don't get a dollar value of the difference. Of course the previous argument gave us two or three different dollar values.

[2] I don't know why the term "deadweight" is used. Musgrave [11] uses the term "excess burden" in 1959. The term excess burden dates back at least to Joseph [6] in 1939, who claims the concept was known to Marshall [8, 8th edition] in 1890. Harberger [3] uses the term "deadweight loss" in 1964, and claims the analysis of the concept goes back at least to Dupuit [2] in 1844.

14.7 Quasilinear utility

A utility function u of $n+1$ goods is **quasilinear** if u is of the form

$$u(x_1, \dots, x_n, y) = y + f(x_1, \dots, x_n),$$

where f is concave and monotone.

- Indifference curves are vertical translates of each other.

- If y is required to be nonnegative, then typically if income is too small, consumption of y will be zero. Once a target level of income is achieved (which may depend on the price vector), then all additional income is spent on good y. (That is, $\dfrac{\partial y*}{\partial m} = \frac{1}{p_y}$.

- The good y is usually taken to be the numéraire.

- If income is such that $y^* > 0$, then, the Hicksian and ordinary demand functions for x_i agree.

14.8 Recovering utility from demand: A little motivation

It is possible to solve differential equations to recover a utility function from a demand function. The general approach may be found in Samuelson [15, 16], but the following discussion is based on Hurwicz and Uzawa [5].

Consider a **demand function**

$$x^* \colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++} \to \boldsymbol{R}^{\mathrm{n}}_{+}$$

derived by maximizing a locally nonsatiated utility function u. Let v be the **indirect utility**, that is,

$$\mathrm{v}(p, m) = u\Big(x^*(p, m)\Big).$$

Since u is locally nonsatiated, the indirect utility function v is strictly increasing in m. The Hicksian **expenditure function** e is defined by

$$e(p, \upsilon) = \min\{p \cdot x : u(x) \geqslant \upsilon\}.$$

We know from the support function theorem that

$$\frac{\partial e(p, \upsilon)}{\partial p_i} = \tilde{x}_i(p, \upsilon) = x_i^*\Big(p, e(p, \upsilon)\Big).$$

Ignoring υ for the moment, we have the **total differential equation**

$$e'(p) = x^*\Big(p, e(p)\Big). \tag{5}$$

What does it mean to solve such an equation? And what happened to υ?

An aside on solutions of differential equations

You may recall from your calculus classes that, in general, differential equations have many solutions, often indexed by "constants of integration." For instance, take the simplest differential equation,

$$y' = a$$

for some constant a. The general form of the solution is

$$y(x) = ax + C,$$

where C is an arbitrary constant of integration. What this means is that the differential equation $y' = a$ has infinitely many solutions, one for each value of C. The parameter υ in our problem can be likened to a constant of integration.

You should also recall that we rarely specify C directly as a condition of the problem, since we don't know the function y in advance. Instead we usually use an **initial condition** (x^0, y^0). That is, we specify that

$$y(x^0) = y^0.$$

In this simple case, the way to translate an initial condition into a constant of integration is to solve the equation

$$y^0 = ax^0 + C \implies C = y^0 - ax^0,$$

and rewrite the solution as

$$y(x) = ax + (y^0 - ax^0) = y^0 + a(x - x^0).$$

In order to make it really explicit that the solution depends on the initial conditions, differential equations texts may go so far as to write the solution as

$$y(x; x^0, y^0) = y^0 + a(x - x^0).$$

In our differential equation (5), an initial condition corresponding to the "constant of integration" υ is a pair (p^0, m^0) satisfying

$$e(p^0, \upsilon) = m^0.$$

From the equivalence of expenditure minimization an utility maximization under a budget constraint, this gives us the relation

$$\upsilon = \mathrm{v}(p^0, m^0) = u\Big(x^*(p^0, m^0)\Big).$$

Following Hurwicz and Uzawa [5], define the **income compensation function** in terms of the Hicksian expenditure function e via[3]

[3] In terms of preferences,

$$\mu(p; p^0, m^0) = \inf\{p \cdot x : x \succcurlyeq x^*(p^0, m^0)\}.$$

Lionel McKenzie [10] employs a similar construction to replace the expenditure function in a framework where only preferences were used, not a utility index. He defines a slightly different function $\mu(p; x^0) = \inf\{p \cdot x : x \succcurlyeq x^0\}$.

$$\mu(p; p^0, m^0) = e\Big(p, \mathrm{v}(p^0, m^0)\Big).$$

Observe that

$$\mu(p^0; p^0, m^0) = m^0$$

and

$$\frac{\partial \mu(p; p^0, m^0)}{\partial p_i} = \frac{\partial e(p, \upsilon^0)}{\partial p_i} = \tilde{x}_i(p, \upsilon^0) = x_i^*\Big(p, e(p, \upsilon^0)\Big) = x_i^*\Big(p, \mu(p; p^0, m^0)\Big).$$

In other words the income compensation function gives the solution $e(p) = \mu(p; p^0, m^0)$ to differential equation (5) in terms of the initial condition $e(p^0) = m^0$.

We are now going to turn the income compensation function around and treat (p^0, m^0) as the variable of interest. Fix a price, any price, $p^* \in \boldsymbol{R}^{\mathrm{n}}_{++}$ and define the function $w \colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++} \to \boldsymbol{R}$ by

$$w(p, m) = \mu(p^*; p, m) = e\Big(p^*, \mathrm{v}(p, m)\Big).$$

The function w is another indirect utility. That is,

$$w(p, m) \geqslant w(p', m') \iff \mathrm{v}(p, m) \geqslant \mathrm{v}(p', m').$$

To see this, observe that since e is strictly increasing in υ,

$$w(p, m) = e\Big(p^*, \mathrm{v}(p, m)\Big) \geqslant e\Big(p^*, \mathrm{v}(p', m')\Big) = w(p', m') \qquad \iff \qquad \mathrm{v}(p, m) \geqslant \mathrm{v}(p', m').$$

We can use w to find a utility U, at least on the range of x^* by

$$U(x) = \mu(p^*; p, m) \qquad \text{where } x = x^*(p, m).$$

14.9 Recovering utility from demand: The plan

The discussion suggests the following approach. Given a demand function x^*:

1. Somehow solve the differential equation

$$\frac{\partial \mu(p)}{\partial p_i} = x_i^*\Big(p, \mu(p)\Big).$$

Write the solution explicitly in terms of the intial condition $\mu(p^0) = m^0$ as $\mu(p; p^0, m^0)$.

2. Use the function μ to define an indirect utility function w by

$$w(p, m) = \mu(p^*; p, m).$$

3. Invert the demand function to give (p, m) as a function of x^*.

4. Define the utility on the range of x^* by

$$U(x) = \mu(p^*; p, m) \quad \text{where } x = x^*(p, m).$$

This is easier said than done, and there remain a few questions. For instance, how do we know that the differential equation has a solution? If a solution exists, how do we know that the "utility" U so derived generates the demand function x^*? We shall address these questions presently, but I find it helps to look at some examples first.

14.10 Examples

In order to draw pictures, I will consider two goods x and y. By homogeneity of degree zero of the demand x^*, I may take good y as numéraire and fix $p_y = 1$, so the price of x will simply be denoted p.

14.10.1 Deriving the income compensation function from a utility

For the Cobb–Douglas utility function

$$u(x, y) = x^\alpha y^\beta$$

where $\alpha + \beta = 1$, the demand functions are

$$x^*(p, m) = \frac{\alpha m}{p}, \quad y^*(p, m) = \beta m.$$

The indirect utility is thus

$$\mathrm{v}(p, m) = m\beta^\beta \left(\frac{\alpha}{p}\right)^\alpha.$$

The expenditure function is

$$e(p, v) = v\beta^{-\beta} \left(\frac{p}{\alpha}\right)^\alpha.$$

Now pick (p^0, m^0) and define

$$\begin{aligned}\mu(p; p^0, m^0) &= e\Big(p; \mathrm{v}(p^0, m^0)\Big) \\ &= \left(m^0\beta^\beta \left(\frac{\alpha}{p^0}\right)^\alpha\right) \beta^{-\beta} \left(\frac{p}{\alpha}\right)^\alpha \\ &= m^0 \left(\frac{p}{p^0}\right)^\alpha.\end{aligned}$$

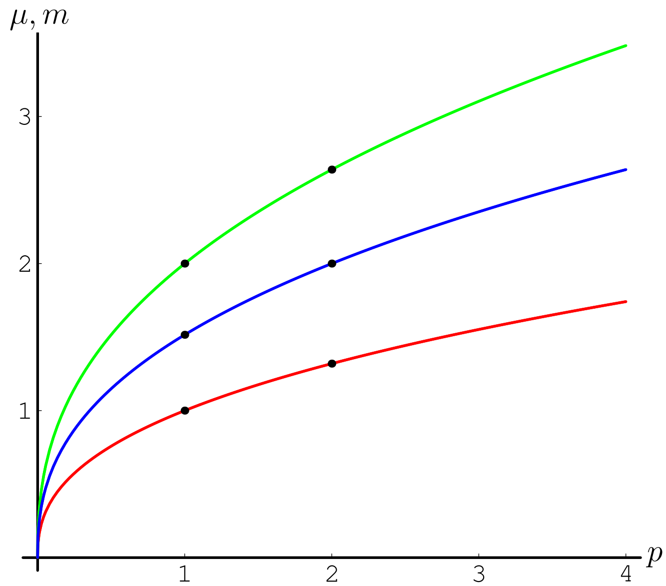

Figure 14.2. Graph of $\mu(p; p^0; m^0)$ for Cobb–Douglas $\alpha = 2/5$ utility and various values of (p^0, m^0).

Evaluating this at $p = p^0$ we have

$$\mu(p^0; p^0, m^0) = m^0.$$

That is, the point (p^0, m^0) lies on the graph of $\mu(\cdot; p^0, m^0)$. Figure 14.2 shows the graph of this function for different values of (p^0, m^0). For each fixed (p^0, m^0), the function $\mu(p) = \mu(p; p^0, m^0)$ satisfies the (ordinary) differential equation

$$\frac{d\mu}{dp} = \alpha \left[m^0 (p^0)^{-\alpha} \right] p^{\alpha-1} = \frac{\alpha \mu(p)}{p} = x^*\Big(p, \mu(p)\Big).$$

Note that homogeneity and budget exhaustion have allowed us to reduce the dimensionality by 1. We have $n-1$ prices, as we have chosen a numéraire, and the demand for the n^{th} good is gotten from $x_n^* = m - \sum_{i=1}^{n-1} p_i x_i^*$.

14.10.2 Examples of recovering utility from demand

Let $n = 2$, and set $p_2 = 1$, so that there is effectively only one price p, and only one differential equation (for x_1)

$$\mu'(p) = x\Big(p, \mu(p)\Big).$$

14.10.1 Example In this example

$$x(p, m) = \frac{\alpha m}{p}.$$

(This x is the demand for x_1. From the budget constraint we can infer $x_2 = (1-\alpha)m$.)

The corresponding differential equation is

$$\mu' = \frac{\alpha\mu}{p} \quad \text{or} \quad \frac{\mu'}{\mu} = \frac{\alpha}{p}.$$

(For those of you more comfortable with y-x notation, this is $y' = \alpha y/x$.) Integrate both sides of the second form to get

$$\ln \mu = \alpha \ln p + C$$

so exponentiating each side gives

$$\mu(p) = Kp^\alpha$$

where $K = \exp(C)$ is a constant of integration. Given the initial condition (p^0, m^0), we must have

$$m^0 = K(p^0)^\alpha, \qquad \text{so } K = \frac{m^0}{(p^0)^\alpha},$$

or

$$\mu(p; p^0, m^0) = \frac{m^0}{(p^0)^\alpha} p^\alpha.$$

For convenience set $p^* = 1$, to get

$$w(p,m) = \mu(p^*; p, m) = \frac{m}{p^\alpha}.$$

To recover the utility u, we need to invert the demand function, that is, we need to know for what budget (p,m) is (x_1,x_2) chosen. The demand function is

$$x_1 = \frac{\alpha m}{p}, \quad x_2 = (1-\alpha)m,$$

so solving for m and p, we have

$$m = \frac{x_2}{1-\alpha}$$

$$x_1 = \frac{\alpha \frac{x_2}{1-\alpha}}{p} \qquad \Longrightarrow \qquad p = \frac{\alpha}{1-\alpha}\frac{x_2}{x_1}.$$

Thus

$$\begin{aligned} u(x_1,x_2) &= w(p,m) \\ &= w\left(\frac{\alpha}{1-\alpha}\frac{x_2}{x_1}, \frac{x_2}{1-\alpha}\right) \\ &= \frac{\frac{x_2}{1-\alpha}}{\left(\frac{\alpha}{1-\alpha}\frac{x_2}{x_1}\right)^\alpha} \\ &= \left(\frac{x_2}{1-\alpha}\right)^{1-\alpha}\left(\frac{x_1}{\alpha}\right)^\alpha \\ &= c x_1^\alpha x_2^{1-\alpha}, \end{aligned}$$

where $c = (1-\alpha)^{1-\alpha}\alpha^\alpha$, which is a Cobb–Douglas utility. □

14.10.2 Example In this example we find a utility that generates a linear demand for x. That is,

$$x(p,m) = \beta - \alpha p.$$

(Note the lack of m.) The differential equation is

$$\mu' = \beta - \alpha p.$$

This differential equation is easy to solve:

$$\mu(p) = \beta p - \frac{\alpha}{2}p^2 + C$$

For initial condition (p^0, m^0) we must choose $C = m^0 - \beta p^0 + \frac{\alpha}{2}{p^0}^2$, so the solution becomes

$$\mu(p; p^0, m^0) = \beta p - \frac{\alpha}{2}p^2 + m^0 - \beta p^0 + \frac{\alpha}{2}{p^0}^2.$$

So choosing $p^* = 0$ (not really allowed, but it works in this case), we have

$$w(p,m) = \mu(p^*; p, m) = m - \beta p + \frac{\alpha}{2}p^2.$$

Given (x, y) (let's use this rather than (x_1, x_2)), we need to find the (p, m) at which it is chosen. We know

$$x = \beta - \alpha p, \quad y = m - px = m - \beta p + \alpha p^2,$$

so

$$p = \frac{\beta - x}{\alpha}, \quad m = y + \beta p - \alpha p^2 = y + \beta\frac{\beta - x}{\alpha} - \alpha\left(\frac{\beta - x}{\alpha}\right)^2.$$

Therefore

$$\begin{aligned} u(x,y) = w(p,m) &= w\left(\frac{\beta - x}{\alpha}, y + \beta\frac{\beta - x}{\alpha} - \alpha\left(\frac{\beta - x}{\alpha}\right)^2\right) \\ &= \underbrace{y + \beta\frac{\beta - x}{\alpha} - \alpha\left(\frac{\beta - x}{\alpha}\right)^2}_{m} - \beta\underbrace{\frac{\beta - x}{\alpha}}_{p} + \frac{\alpha}{2}\underbrace{\left(\frac{\beta - x}{\alpha}\right)^2}_{p^2} \\ &= y - \frac{(\beta - x)^2}{2\alpha}. \end{aligned}$$

Note that the utility is decreasing in x for $x > \beta$. Representative indifference curves are shown in Figure 14.3. The demand curve specified implies that x and y will be negative for some values of p and m, so we can't expect that this is a complete specification. I'll leave it to you to figure out when this makes sense. □

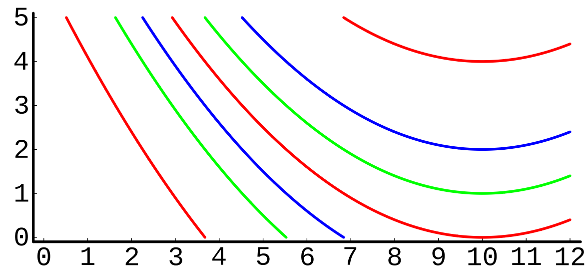

Figure 14.3. Indifference curves for Example 14.10.2 (linear demand) with $\beta = 10$, $\alpha = 5$.

14.11 A general integrability theorem

Hurwicz and Uzawa [5] prove the following theorem, presented here without proof.

14.11.1 Hurwicz–Uzawa Integrability Theorem *Let* $\xi\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}^{\mathrm{n}}_{+}$. *Assume*

(B) *The budget exhaustion condition*

$$p \cdot \xi(p, m) = m$$

is satisfied for every $(p, m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$.

(D) *Each component function* ξ_i *is differentiable everywhere on* $\boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$.

(S) *The Slutsky matrix is symmetric, that is, for every* $(p, m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$,

$$S_{i,j}(p, m) = S_{j,i}(p, m) \qquad i, j = 1, \dots, n.$$

(NSD) *The Slutsky matrix is negative semidefinite, that is, for every* $(p, m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$, *and every* $v \in \boldsymbol{R}^{\mathrm{n}}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} S_{i,j}(p, m) v_i v_j \leqslant 0.$$

(IB) *The function* ξ *satisfies the following boundedness condition on the partial derivative with respect to income. For every* $0 \ll \underline{a} \ll \bar{a} \in \boldsymbol{R}^{\mathrm{n}}_{++}$, *there exists a (finite) real number* $M_{\underline{a},\bar{a}}$ *such that for all* $m \geqslant 0$

$$\underline{a} \leqq p \leqq \bar{a} \quad \Longrightarrow \quad \left| \frac{\partial \xi_i(p, m)}{\partial m} \right| \leqslant M_{\underline{a},\bar{a}} \quad i = 1, \dots, n.$$

Let X *denote the range of* ξ,

$$X = \{\xi(p, m) \in \boldsymbol{R}^{\mathrm{n}}_{+} : (p, m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}\}.$$

Then there exists a utility function $u\colon X \to \boldsymbol{R}$ *on the range* X *such that for each* $(p, m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$,

$\xi(p, m)$ *is the unique maximizer of* u *over the budget set* $\{x \in X : p \cdot x \leqslant m\}$.

14.12 Samuelson's Weak Axiom of Revealed Preference

The Weak Axiom of Revealed Preference asserts that if you demand x when y is in the budget set, it is because you prefer x to y. Therefore you should never demand y when x is in the budget set. (This of course implicitly assumes a unique utility maximizer, or strict quasiconcavity of the utility.) Paul Samuelson [12, 13, 14, 17] showed that this observation alone is enough to deduce the negative semidefiniteness of the matrix of Slutsky substitution terms.

14.12.1 Definition (Samuelson's Weak Axiom of Revealed Preference) *Let $X \subset \boldsymbol{R}^{\mathrm{n}}$ be the consumption set. For an ordinary demand function $x^*\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++} \to X$, define the binary relation S on X by*

$$x \; S \; y \quad \text{if} \quad (\,\exists (p,w)\,)\;[\,x = x^*(p,w) \;\&\; y \neq x \;\&\; p \cdot y \leqslant w\,].$$

That is, x is demanded when y is in the budget set but not demanded, so x is ***revealed preferred*** *to y. The demand function x^* obeys* ***Samuelson's Weak Axiom of Revealed Preference (SWARP)*** *if S is an asymmetric relation. That is, if for every $x, y \in X$,*

$$x \; S \; y \implies \neg y \; S \; x.$$

That is, if x is revealed preferred to y, then y is never revealed preferred to x.

The demand function x^* satisfies the **budget exhaustion condition** if for all (p, w),

$$p \cdot x^*(p,w) = w.$$

Under the budget exhaustion condition, we can rewrite SWARP in the form that Samuelson used. Let x^0 and x^1 belong to the range of x^*. That is, let

$$x^0 = x^*(p^0, w^0) = x^*(p^0, p^0 \cdot x^0) \quad \text{and} \quad x^1 = x^*(p^1, w^1) = x^*(p^1, p^1 \cdot x^1).$$

Then $p^1 \cdot x^0 \leqslant p^1 \cdot x^1$ and $x^0 \neq x^1$ imply $x^1 \; S \; x^0$; while $x^0 \neq x^1$ and $\neg x^0 \; S \; x^1$ imply $p^0 \cdot x^1 > p^0 \cdot x^0$. Thus, we can write SWARP in Samuelson's form:[4]

$$x^0 \neq x^1 \text{ and } p^1 \cdot x^0 \leqslant p^1 \cdot x^1 \quad \implies \quad p^0 \cdot x^1 > p^0 \cdot x^0.$$

14.13 Slutsky compensated demand

This leads us to define the **Slutsky compensated demand** s in terms of the ordinary demand function x^* via

$$s(p, \bar{x}) = x^*(p, p \cdot \bar{x})$$

[4] It may appear that this condition is weaker then than the one stated above, since it applies only to x_0 and x_1 in the range of x^*, whereas the condition above applies to all x and y in X, which may be larger than the range of x^*. However, any violation of SWARP as stated above would involve x and y with $x \; S \; y$ and $y \; S \; x$, which can only happen if both x and y belong to the range of x^*. Thus the definitions are equivalent.

where $\bar{x} \in X$ can be thought of as an initial endowment that determines the value of income w. Another interpretation is that if $\bar{x} = x^*(\bar{p}, \bar{w})$, then $s(p, \bar{x})$ is the demand $x^*(p, w)$ where w has been adjusted (compensated) so that consumption $\bar{x}$ is still just affordable at price vector p.

Note that
$$\frac{\partial s_i(p, \bar{x})}{\partial p_j} = \frac{\partial x_i^*(p, p \cdot \bar{x})}{\partial p_j} + \bar{x}_j \frac{\partial x_i^*(p, p \cdot \bar{x})}{\partial w}.$$
In particular, by setting $\bar{x} = x^*(p, w)$ we may define the **Slutsky substitution term**

$$\begin{aligned}\sigma_{i,j}(p, w) &= \frac{\partial s_i\Big(p, x^*(p, w)\Big)}{\partial p_j} \\ &= \frac{\partial x_i^*(p, w)}{\partial p_j} + x_j^*(p, w) \frac{\partial x_i^*(p, w)}{\partial w}.\end{aligned}$$

The following important lemma may be found in Samuelson [17, equation (70), p. 109] or Mas-Colell, Whinston, and Green [9, Proposition 2.F.1, pp. 30–33].

14.13.1 Lemma *Let x^* satisfy the budget exhaustion condition and SWARP. Let*
$$x^0 = x^*(p^0, w^0) \quad \textit{and} \quad x^1 = x^*(p^1, p^1 \cdot x^0).$$
Then
$$(p^1 - p^0) \cdot (x^1 - x^0) \leqslant 0,$$
with equality if and only if $x^1 = x^0$.

Proof: If $x^1 = x^0$, then the conclusion is true as an equality. So assume $x^1 \neq x^0$.

By budget exhaustion
$$p^1 \cdot x^1 = p^1 \cdot x^0. \tag{6}$$
Since $x^1 \neq x^0$, this says that $x^1 \; S \; x^0$. So by SWARP, we have $\neg x^0 \; S \; x^1$, that is,
$$p^0 \cdot x^1 > w^0 = p^0 \cdot x^0, \tag{7}$$
where the second equality follows from budget exhaustion. Subtracting inequality (7) from equality (6) gives
$$(p^1 - p^0) \cdot x^1 < (p^1 - p^0) \cdot x^0,$$
which proves the conclusion of the lemma. ∎

14.13.2 Theorem *Let $x^* \colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++} \to \boldsymbol{R}^{\mathrm{n}}_{+}$ be differentiable and satisfy the budget exhaustion condition and SWARP. Then for every $(p, w) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++}$, and every $v \in \boldsymbol{R}^{\mathrm{n}}$,*
$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j}(p, w) v_i v_j \leqslant 0.$$

That is, the matrix of Slutsky substitution terms is negative semidefinite.[5]

Proof: Fix $(p, w) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++}$ and $v \in \boldsymbol{R}^{\mathrm{n}}$. By homogeneity of degree 2 of the quadratic form in v, without loss of generality we may scale v so that $p \pm v \gg 0$.

Define the function x on $[-1, 1]$ via

$$x(t) = s\Big(p + tv, x^*(p, w)\Big). \tag{8}$$

Note that this is differentiable, and $x(0) = x^*(p, w)$.

By Lemma 14.13.1 (with $p + tv$ playing the rôle of p^1 and p playing the rôle of p^0),

$$(p + tv - p) \cdot \Big(x(t) - x(0)\Big) = tv \cdot \Big(x(t) - x(0)\Big) \leqslant 0.$$

For nonzero t, dividing by $t^2 > 0$ gives

$$v \cdot \frac{x(t) - x(0)}{t} \leqslant 0.$$

Taking limits as $t \to 0$ gives

$$v \cdot x'(0) \leqslant 0. \tag{9}$$

By the Chain Rule applied to (8),

$$x_i'(t) = \sum_{j=1}^{n} \frac{\partial s_i\Big(p + tv, x^*(p, w)\Big)}{\partial p_j} v_j. \tag{10}$$

Evaluating (10) at $t = 0$ yields

$$\begin{aligned} x_i'(0) &= \sum_{j=1}^{n} \frac{\partial s_i\Big(p, x^*(p, w)\Big)}{\partial p_j} v_j \\ &= \sum_{j=1}^{n} \sigma_{i,j}(p, w) v_j, \end{aligned}$$

where the second equality is just the definition of $\sigma_{i,j}(p, w)$. Combining this with (9) gives

$$0 \geqslant v \cdot x'(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j}(p, w) v_i v_j,$$

which completes the proof. ∎

This proof is Kihlstrom, Mas-Colell, and Sonnenschein's [7] modern rewriting of Samuelson's argument.

[5] Most authors, myself included, usually reserve the term "negative semidefinite" for *symmetric* matrices. In this instance, I won't insist on it.

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