Ec 121a
Theory of Value

KC Border
Fall 2020

Lecture 13: Demand Theory 2

13.1 Duality, the Envelope Theorem, and Demand

The following lengthy table summarizes the properties of the solutions to the utility maximization and expenditure minimization problems.

Utility Maximization	Expenditure Minimization
$\underset{x}{\text{maximize}}\, u(x)$ subject to $m - p \cdot x \geqslant 0$	$\underset{x}{\text{minimize}}\, p \cdot x$ subject to $u(x) - v \geqslant 0$
Optimal Solutions	
Ordinary (Walrasian) Demand	Hicksian Compensated Demand
$x^*(p, m)$	$\hat{x}(p, v)$
x^* is homogeneous of degree zero in (p, m).	
Optimal Value functions	
Indirect Utility	Expenditure Function
$\mathrm{v}(p, m) = u\Big(x^*(p, m)\Big)$	$e(p, v) = p \cdot \hat{x}(p, v)$
v is quasiconvex in (p, m), homogeneous of degree zero in (p, m).	e is concave in p, and homogeneous of degree 1 in p.
Statement of Equivalence (Theorem 12.10.1)	
$x^*(p, m) = \hat{x}\Big(p, \mathrm{v}(p, m)\Big)$ $m = e\Big(p, \mathrm{v}(p, m)\Big)$	$\hat{x}(p, v) = x^*\Big(p, e(p, v)\Big)$ $v = \mathrm{v}\Big(p, e(p, v)\Big)$

Utility Maximization	**Expenditure Minimization**
Lagrangeans	
$L(x, \lambda; p, m) = u(x) + \lambda(m - p \cdot x)$	$L(x, \mu; p, \upsilon) = p \cdot x - \mu\Big(u(x) - \upsilon\Big)$
Partials with respect to parameters	
$\dfrac{\partial L(x, \lambda; p, m)}{\partial p_j} = -\lambda x_j$	$\dfrac{\partial L(x, \mu; p, \upsilon)}{\partial p_j} = x_j$
$\dfrac{\partial L(x, \lambda; p, m)}{\partial m} = \lambda$	$\dfrac{\partial L(x, \mu; p, \upsilon)}{\partial \upsilon} = \mu$
Envelope Theorem	
$\dfrac{\partial \mathrm{v}(p, m)}{\partial p_j} = -\lambda^*(p, m) x_j^*(p, m)$	$\dfrac{\partial e(p, \upsilon)}{\partial p_j} = \hat{x}_j(p, \upsilon)$
$\dfrac{\partial \mathrm{v}(p, m)}{m} = \lambda^*(p, m)$	$\dfrac{\partial e(p, \upsilon)}{\partial \upsilon} = \hat{\mu}(p, \upsilon)$
Roy's Law	Hotelling/Shephard's Lemma
$x_j^*(p, m) = -\dfrac{\dfrac{\partial \mathrm{v}(p, m)}{\partial p_j}}{\dfrac{\partial \mathrm{v}(p, m)}{\partial m}}$	$\hat{x}_j(p, \upsilon) = \dfrac{\partial e(p, \upsilon)}{\partial p_j}$

Differentiating the equivalence $m = e\Big(p, \mathrm{v}(p, m)\Big)$ with respect to m yields

$$1 = \frac{\partial e\Big(p, \mathrm{v}(p, m)\Big)}{\partial \upsilon} \frac{\partial \mathrm{v}(p, m)}{\partial m} = \hat{\mu}\Big(p, \mathrm{v}(p, m)\Big)\lambda^*(p, m),$$

or equivalently,

$$\hat{\mu}\Big(p, \mathrm{v}(p, m)\Big) = \frac{1}{\lambda^*(p, m)} \quad \text{and} \quad \hat{\mu}(p, \upsilon) = \frac{1}{\lambda^*\Big(p, e(p, \upsilon)\Big)}.$$

From the equivalence

$$\hat{x}(p, \upsilon) = x^*\Big(p, e(p, \upsilon)\Big)$$

we have

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial x_i^*\Big(p, e(p,\upsilon)\Big)}{\partial p_j} + \frac{\partial x_i^*\Big(p, e(p,\upsilon)\Big)}{\partial m}\frac{\partial e(p,\upsilon)}{\partial p_j}.$$

But $\frac{\partial e(p,\upsilon)}{\partial p_j} = \hat{x}_j(p,\upsilon) = x_j^*\Big(p, e(p,\upsilon)\Big)$. Set $m = e(p,\upsilon)$, and write

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial x_i^*(p,m)}{\partial p_j} + x_j^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m}$$

which implies the **Slutsky equation**

$$\boxed{\frac{\partial x_i^*(p,m)}{\partial p_j} = \frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} - x_j^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m},}$$

where $\upsilon = \mathrm{v}(p,m)$, which decomposes the effect of a price change into its **substitution effect** and **income effect**.

But

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial^2 e(p,\upsilon)}{\partial p_i \partial p_j},$$

so since e is concave in p, its Hessian is negative semidefinite (and symmetric), so the matrix

$$\left[\frac{\partial x_i^*(p,m)}{\partial p_j} + x_j^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m}\right] \text{ is negative semidefinite and symmetric.}$$

Consequently the diagonal terms satisfy

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_i} = \frac{\partial x_i^*(p,m)}{\partial p_i} + x_i^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m} \leqslant 0,$$

and we have the unusual **reciprocity** relation

$$\frac{\partial x_i^*(p,m)}{\partial p_j} + x_j^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m} = \frac{\partial x_j^*(p,m)}{\partial p_i} + x_i^*(p,m)\frac{\partial x_j^*(p,m)}{\partial m}.$$

13.2 Axioms for a numerical standard of living measure

Assume n "commodities," some of which may be interpreted as instruments of saving. A **standard of living index**

$$\mathrm{v}\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}$$

assigns a real number to each **budget**[1] $(p,m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$, where p is a vector of prices and m is money income. The following are natural requirements for a standard of living measure.

[1] I shall use the term budget to refer to both the pair (p,m) and the budget set $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : p \cdot x \leqslant m\}$. You should not get confused.

Monotonicity in m. $m' > m \implies \mathrm{v}(p, m') > \mathrm{v}(p, m)$.

Note that the strict monotonicity suggests a kind of nonsatiation.

Monotonicity in p. $(m > 0 \;\&\; p' \gg p) \implies \mathrm{v}(p', m) < \mathrm{v}(p, m)$.[2 3]

By requiring $p' \gg p$ we are allowing for the possibility that some commodities are irrelevant.

Homogeneity. $\mathrm{v}(p, m)$ is homogeneous of degree zero in (p, m).

Another property that a standard of living measure ought to incorporate what I shall call the **Lancaster Principle**. Kelvin Lancaster [4, p. 65] argued that

> *An individual's welfare has unambiguously increased from situation I to situation II if his choice is expanded as a result of the change, that is, if, in situation II he can have—*
>
> 1. *what he chose in situation I, and*
> 2. *at least one choice not available to him in situation I.*

I argue that this principle implies the standard of living measure v has the following property. (Compare this to the proof of Proposition 12.11.1.)

Quasiconvexity in p. For any $p^0, p^1 \in \boldsymbol{R}^{\mathrm{n}}_{++}$, $m \geqslant 0$, λ satisfying $0 < \lambda < 1$,

$$\mathrm{v}\Big((1-\lambda)p^0 + \lambda p^1, m\Big) \leqslant \max\Big\{\mathrm{v}(p^0, m), \mathrm{v}(p^1, m)\Big\}.$$

To see why this captures the Lancaster Principle, let $p^\lambda = (1-\lambda)p^0 + \lambda p^1$, where $0 < \lambda < 1$, and let x^λ be purchased in budget (p^λ, m). It must be the case that either $p^0 \cdot x^\lambda \leqslant m$ or $p^1 \cdot x^\lambda \leqslant m$ or both. (Otherwise we would have $p^0 \cdot x^\lambda > m$ and $p^1 \cdot x^\lambda > m$, so the convex combination of these two yields $p^\lambda \cdot x^\lambda > m$, which contradicts the fact that x^λ is affordable at (p^λ, m).) Now, unless we have one of the trivial cases $p^0 = p^\lambda = p^1$ or $m = 0$, one of the budgets

[2] Use this notation for vector orders:

$$x \geqq y \text{ if } x_i \geqslant y_i \text{ for all } i; \qquad x \gg y \text{ if } x_i > y_i \text{ for all } i;$$

and define

$$\boldsymbol{R}^{\mathrm{n}}_{+} = \{x : x \geqq 0\}; \qquad \boldsymbol{R}^{\mathrm{n}}_{++} = \{x : x \gg 0\}.$$

[3] A function u is **monotone** if

$$x' \gg x \quad \implies \quad u(x') > u(x).$$

(p^0, m) or (p^1, m) includes x^λ plus choices *not* available in budget (p^λ, m).[4] Thus Lancaster's Principle implies $\mathrm{v}(p^\lambda, m) \leqslant \mathrm{v}(p^0, m)$, and quasiconvexity is assured.

We now show that with one technical assumption (upper semicontinuity in income) any standard of living index satisfying the properties above must be the indirect utility for some utility function.

13.3 The indirect utility problem

It is well known know that the indirect utility v derived from a locally nonsatiated continuous utility u will be continuous and satisfy properties **P**–**S** below. (Property **N** is a harmless normalization.) See, e.g., Diewert [2], Varian [6, Section 7.3], Mas-Colell, Whinston, and Green [5, Prop. 3.D.3]. Moreover u is determined on the range of the demand function by the inversion formula in the next lemma.

13.3.1 Lemma *If* v *is the indirect utility for a locally nonsatiated utility u, then for any x in the range of the demand function,*

$$u(x) = \min_p \mathrm{v}(p, p \cdot x).$$

Proof: Fix $\bar{x} = x^*(\bar{p}, \bar{m})$ in the range of the demand function. For any p, the point $\bar{x}$ belongs to the budget $(p, p \cdot \bar{x})$, so $u(\bar{x}) \leqslant \mathrm{v}(p, p \cdot \bar{x})$. But for $\bar{p}$, we have $u(\bar{x}) = \mathrm{v}(\bar{p}, \bar{m}) = \mathrm{v}(\bar{p}, \bar{p} \cdot \bar{x})$ by budget exhaustion. q.e.d. ∎

The obvious question is: If v satisfies properties **N**–**S** is it an indirect utility? Surprisingly, the standard texts mentioned above do not answer this question. The inversion formula determines what u ought to be, but it remains to prove that v is the maximized value of u over the budget set. Diewert [2, Theorem 4, p. 558] comes close to stating Theorem 13.3.2, but he makes the ad hoc assumption the function u derived by the inversion formula is continuous. In general it is not. Krishna and Sonnenschein [3] prove Theorem 13.3.2 by different means.

13.3.2 Indirect Utility Theorem *Let* $\mathrm{v} \colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_+ \to \boldsymbol{R}$ *satisfy the following properties:*

N *(Nonnegativity):* $\mathrm{v}(p, m) \geqslant 0$ *for all* (p, m).

P *(Monotonicity in p):* $(m > 0 \;\&\; p' \gg p) \implies \mathrm{v}(p', m) < \mathrm{v}(p, m)$.

M *(Monotonicity in m):* $m' > m \implies \mathrm{v}(p, m') > \mathrm{v}(p, m)$.

H *(Homogeneity):* $\mathrm{v}(\lambda p, \lambda m) = \mathrm{v}(p, m)$ *for all* $\lambda > 0$.

[4] Unless $p^0 \geqq p^1$ or vice versa, each of p^0 and p^1 will have at least one commodity price lower than p^λ and so offer new choices. If say $p^1 \geqq p^0$ and $p^1 \neq p^0$, then it must be that x^λ is affordable at p^0, which has at least one commodity price lower than p^λ, so it offers new choices.

Q *(Quasiconvexity in p):* $\mathrm{v}(p,m)$ *is quasiconvex in p.*

S *(Upper semicontinuity in m):* $\mathrm{v}(p,\cdot)$ *is upper semicontinuous for each p.*

Then there is an upper semicontinuous monotone quasiconcave utility $u\colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$ *such that*

$$\mathrm{v}(p,m) = \max\Big\{u(x) : x \in \boldsymbol{R}^{\mathrm{n}}_{+} \text{ and } p \cdot x \leqslant m\Big\}.$$

Moreover, for any x in the range of the demand generated by u, we have

$$u(x) = \min_{p} \mathrm{v}(p, p \cdot x).$$

The proof is given in Section 13.5.

This theorem leaves one loose end. Suppose I start with a utility $\hat{u}$ having indirect utility v. Theorem 13.3.2 asserts that the utility u generated by the inversion formula also has indirect utility v, but Example 13.3.3 below provides an example where u is not equal to $\hat{u}$, so the question remains, do u and $\hat{u}$ generate the same demand correspondence? If v is differentiable, then Roy's Law guarantees that the demand correspondence is actually a function and the demand function is determined by v. But we know that if $\hat{u}$ is not quasiconcave, then the recovered utility u may have a demand correspondence that properly includes the demand of $\hat{u}$, so to hope to prove that the demand depends only on v we need to restrict ourselves to quasiconcave utilities. Theorem 13.3.4 below provides a generalization of Roy's Law that holds even if v is not differentiable.

13.3.3 Example Consider the quasilinear utility function for two goods x and y defined by

$$u(x,y) = y - (1-x)^2$$

which gives a linear demand function for x. It is locally nonsatiated but not monotone. It has the property that the demand for x never exceeds 1. It has the same demand behavior as the monotone utility

$$\hat{u}(x,y) = \begin{cases} y - (1-x)^2 & x \leqslant 1 \\ y & x \geqslant 1 \end{cases}$$

See Figure 13.1. □

In general, you can show the following: Let u be a locally nonsatiated continuous utility with indirect utility v, and let $\hat{u}(x) = \inf_p \mathrm{v}(p, p\cdot x)$. If x^* is in the range of the demand function for u, then $\hat{u}(x^*) = u(x^*)$ and

$$\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : \hat{u}(x) \geqslant \hat{u}(x^*)\} \text{ is the increasing convex hull of } \{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : u(x) \geqslant u(x^*)\}.$$

(The increasing convex hull of a set E is the smallest increasing convex set that includes E. A set E is **increasing** if $x \in E$ and $x' \gg x$ imply $x' \in E$.) I should write down the proof some time. If x is not in the range of the demand, then all bets are off, see Example 13.3.3.

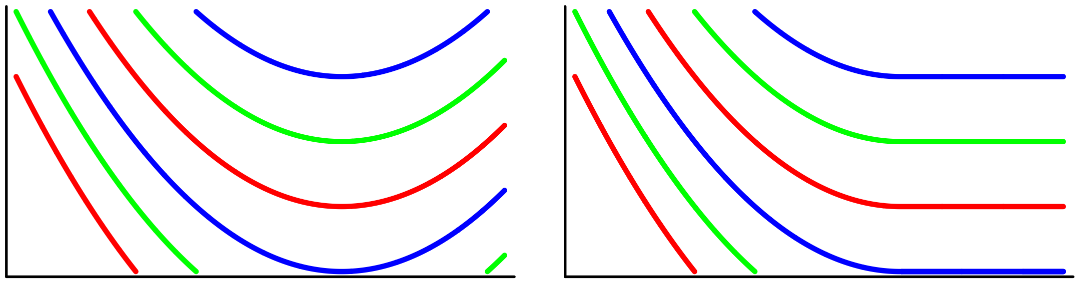

Figure 13.1. Indifference curves for locally nonsatiated utility $u(x, y) = y-(1-x)^2$ and monotone utility with same demand.

A variant of Roy's Law

The proof of Theorem 13.3.2 suggests the following variant of Roy's law, which does not require differentiability. It may be found in Diewert [2, Corollary 4.1, p. 558].

13.3.4 Theorem *Let* v *be an indirect utility function satisfying the hypotheses of Theorem 13.3.2. Let* x^* *be the demand correspondence derived from the recovered utility. Then for* $\bar m > 0$ *and any* $\bar p \in \boldsymbol{R}^{\mathrm{n}}_{++}$,

$$x^*(\bar p, \bar m) = \Big\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : \bar p \cdot x = \bar m \;\&\; x \text{ supports } \{p \in \boldsymbol{R}^{\mathrm{n}}_{++} : \mathrm{v}(p, \bar m) < \mathrm{v}(\bar p, \bar m)\} \text{ at } \bar p\Big\}.$$

Proof: First assume that $\bar x \in x^*(\bar p, \bar m)$. Then $\bar p \cdot \bar x \leqslant \bar m$ and $u(\bar x) = \mathrm{v}(\bar p, \bar m)$. We wish to prove that $\mathrm{v}(p, \bar m) < \mathrm{v}(\bar p, \bar m)$ implies $\bar x \cdot p > \bar x \cdot \bar p$. We shall prove the contrapositive. So assume that $\bar x \cdot p \leqslant \bar x \cdot \bar p \leqslant \bar m$. The $\bar x$ belongs to the budget $(p, \bar m)$, so $\mathrm{v}(p, \bar m) \geqslant u(\bar x) = \mathrm{v}(\bar p, \bar m)$.

For the converse, we have already proven (see (8) in the proof of Theorem 13.3.2) that if x supports the convex set $\{p : \mathrm{v}(p, \bar m) < \mathrm{v}(\bar p, \bar m)\}$ at $\bar p$, then x belongs to the demand set $x^*(\bar p, \bar m)$. ■

13.4 A composite commodity theorem

For the purpose of drawing pictures, I often want to put a single good, say "corn," on one axis, and "everything else" on the other. In this section I show that as long as I don't change the price of any of the other goods, we can find a utility over the two goods corn and the composite good, everything else, that generates the demand for corn.

In fact, there is a much stronger result allows us to partition the set of n goods into N disjoint subsets and as long as the price ratios within a subset are fixed, there is a utility that generates the demand for these N composite commodities.

The proof is the same as for the case here, but the notation is more complicated. For details see my on-line notes [1].

So let u be a utility function for the $n+1$ goods, good x and goods $y_1, \ldots, y_n$. Let

$$(x^*, y^*) = \Big(x^*(p_x, p_1, \ldots, p_n; m), y_1^*(p_x, p_1, \ldots, p_n; m), \ldots, y_n^*(p_x, p_1, \ldots, p_n; m)\Big)$$

be the demand function for these $n+1$ goods and let $\mathrm{v}(p_x, p_1, \ldots, p_n; m)$ be the indirect utility.

I need to work on improving this notation.

Now fix the prices of goods $y_1, \ldots, y_n$ at

$$\overline{p}_Y = (\overline{p}_1, \ldots, \overline{p}_n).$$

We now create a **composite good** Y out of $(y_1, \ldots, y_n)$ by defining

$$Y(y) = \sum_{i=1} \overline{p}_i y_i.$$

Note that the units of this composite good are measured in money. The composite good Y is just the amount spent on the shopping cart y at prices $\bar{p}_Y$. For the composite consumption vector we have a price vector π with only two prices, π_X, the price of X, and π_Y, the price of the composite good Y. Usually the composite good is taken as the numéraire, and some authors refer to the composite good as money.

Define the "demand" for $X = x$ and the composite good Y by

$$\begin{aligned} X^*(\pi_X, \pi_Y; m) &= x^*\Big(\underbrace{(\pi_X, \pi_Y \overline{p}_Y)}_{\in \boldsymbol{R}^{\mathrm{n}+1}}, m\Big) \\ Y^*(\pi_X, \pi_Y; m) &= \sum_{j=1}^{n} \overline{p}_j y_j^*\Big(\underbrace{(\pi_X, \pi_Y \overline{p}_Y)}_{\in \boldsymbol{R}^{\mathrm{n}+1}}, m\Big). \end{aligned} \tag{1}$$

Then we have the following.

13.4.1 A simple composite commodity demand theorem *There is a (quasiconcave, monotonic, upper semicontinuous) utility* $U \colon \boldsymbol{R}^2_+ \to \boldsymbol{R}$ *such that the pair*

$$\Big(X^*(\pi_X, \pi_Y; m), Y^*(\pi_X, \pi_Y; m)\Big)$$

defined by (1) *maximizes* U *over the budget set*

$$\{(X, Y) \in \boldsymbol{R}^2_+ : \pi_X X + \pi_Y Y \leqslant m\}.$$

Proof: The idea is to find an indirect utility and use the Indirect Utility Theorem. But there is an obvious candidate. Define

$$V(\pi_X, \pi_Y; m) = \mathrm{v}(\pi_X, \pi_Y \overline{p}_Y; m).$$

Then V is clearly increasing in m, decreasing in (π_X, π_Y), and has the same continuity properties with respect to m that v does. The only thing that is not immediately obvious is that V is quasiconvex in π. So let $0 < \lambda < 1$. Then

$$\begin{aligned} V\Big((1-\lambda)\pi^1 + \lambda\pi^2; m\Big) &= \mathrm{v}\Big((1-\lambda)\pi_X^1 + \lambda\pi_X^2, \Big((1-\lambda)\pi_Y^1 + \lambda\pi_Y^2\Big)\overline{p}_Y; m) \\ &\leqslant \max\{\mathrm{v}(\pi_X^1, \pi_Y^1\overline{p}_Y; m), \mathrm{v}(\pi_X^2, \pi_Y^2\overline{p}_Y; m)\} \\ &= \max\{V(\pi^1; m), V(\pi^2; m)\}, \end{aligned}$$

where the inequality follows from the quasiconvexity of v. Thus, V is quasiconvex.

We have shown that V is the indirect utility for some utility $U\colon \boldsymbol{R}^2_+ \to \boldsymbol{R}_+$ over the goods X and Y. From the Indirect Utility Theorem, we know that

$$U(X, Y) = \inf_{\pi} V(\pi_X, \pi_Y; \pi_X X + \pi_Y Y)$$

and

$$V(\pi, m) = \max\{U(X, Y) : \pi_X X + \pi_Y Y \leqslant m\}$$

We now have to show that the “demands” X^* and Y^* defined by (1) are generated by maximizing U over the budget set. Since V is the maximized utility, what we need to show is that

$$U\Big(X^*(\pi, m), Y^*(\pi, m)\Big) = V(\pi, m). \tag{2}$$

It becomes notationally easier if we define two mappings:

$$\varphi(\pi_X, \pi_Y) = (\pi_X, \pi_Y\overline{p}_Y),$$

which maps a vector of prices for the composite goods back to a vector of prices for all $n+1$ goods, and

$$\boldsymbol{\xi}(x, y) = (x, \overline{p}_Y \cdot y),$$

which maps a vector of $n+1$ goods down to a two-dimensional composite vector. Observe that

$$\pi \cdot \boldsymbol{\xi}(x, y) = \varphi(\pi) \cdot (x, y). \tag{3}$$

Now fix $(\overline{\pi}, \overline{m})$, and to further simplify notation, let

$$\overline{p} = (\overline{\pi}_X, \overline{\pi}_Y p_Y), \quad x^* = x^*(\overline{p}, \overline{m}), \quad y^* = y^*(\overline{p}, \overline{m}), \quad X^* = X^*(\overline{\pi}, \overline{m}), \quad \text{and} \quad Y^* = Y^*(\overline{\pi}, \overline{m}).$$

Then

$$\begin{aligned} \mathrm{v}(\bar{p}, \bar{m}) &= u(x^*, y^*) \\ &= \inf\nolimits_p \mathrm{v}\Big(p, p \cdot (x^*, y^*)\Big) && \text{Theorem 13.3.2} \\ &\leqslant \inf\nolimits_\pi \mathrm{v}\Big(\varphi(\pi), \varphi(\pi) \cdot (x^*, y^*)\Big) && \text{infimum over a smaller set of prices} \\ &= \inf\nolimits_\pi V\Big(\pi, \pi \cdot \boldsymbol{\xi}(x^*, y^*)\Big) && \text{construction of } V \text{ and equation (3)} \\ &= U\Big(\boldsymbol{\xi}(x^*, y^*)\Big) && \text{construction of } U \\ &= U(X^*, Y^*) && \text{construction of } X^*, Y^* \\ &\leqslant V(\bar{\pi}, \bar{m}) && \text{since } \bar{\pi}_X \cdot X^*(\bar{\pi}, \bar{m}) + \pi_Y Y^*(\bar{\pi}, \bar{m}) \leqslant \bar{m} \\ &= \mathrm{v}\Big(\varphi(\bar{\pi}), \bar{m}\Big) && \text{construction of } V \\ &= \mathrm{v}(\bar{p}, \bar{m}) && \text{construction of } \bar{p}. \end{aligned}$$

Thus all the inequalities are equalities, and we are done, and as a result we can conclude

$$U(X^*, Y^*) = V(\bar{\pi}, \bar{m}),$$

which proves (2), and we are done. ■

13.5 Proof of the Indirect Utility Theorem

Proof of Theorem 13.3.2: The proof is divided into several steps.

Step 0. $\mathrm{v}(p, 0)$ *is independent of* p:
Since $\mathrm{v}(p, \cdot)$ is upper semicontinuous in m and decreasing in m, it is therefore continuous at $m = 0$ for each p. Pick $p, p' \gg 0$. For $\lambda > 0$ large enough, we have $\lambda p \gg p'$. Then for each $n \geqslant 1$,

$$\mathrm{v}(p, 1/n) = \mathrm{v}(\lambda p, \lambda/n) > \mathrm{v}(p', \lambda/n),$$

where the equality follows from **H**, and the inequality follows from **P**. Letting $n \to \infty$, continuity at 0 implies $\mathrm{v}(p, 0) \geqslant \mathrm{v}(p', 0)$. But the roles of p and p' are symmetrical, so the reverse inequality also holds, implying

$$\mathrm{v}(p, 0) = \mathrm{v}(p', 0) \quad \text{for all } p, p' \tag{4}$$

Step 1. Construction of u:
Define u via the inversion formula

$$u(x) = \inf\{\mathrm{v}(p, p \cdot x) : p \in \boldsymbol{R}^{\mathrm{n}}_{++}\}. \tag{$\star$}$$

Property **N** (nonnegativity) guarantees that this infimum is not $-\infty$, and in fact u is a nonnegative real-valued function defined for all $x \geqq 0$. •

Step 2. v *is the indirect utility for* u:
I first claim that

$$\bar{p} \cdot x \leqslant \bar{m} \quad \implies \quad u(x) \leqslant \mathrm{v}(\bar{p}, \bar{m}). \tag{5}$$

To see this note that

$$u(x) = \inf_p \mathrm{v}(p, p \cdot x) \leqslant \mathrm{v}(\bar{p}, \bar{p} \cdot x) \leqslant \mathrm{v}(\bar{p}, \bar{m}),$$

where the last inequality follows from **M** (monotonicity in m).

Inequality (5) shows that

$$\mathrm{v}(\bar{p}, \bar{m}) \geqslant \sup\{u(x) : \bar{p} \cdot x \leqslant \bar{m}\}.$$

We now need to prove that the supremum is attained at some point, and that the inequality holds with equality.

There are two cases to be dealt with, $m = 0$ and $m > 0$. Start with the case $m = 0$. Then

$$\max\{u(x) : x \in \boldsymbol{R}^{\mathrm{n}}_{+} \text{ and } \bar{p} \cdot x \leqslant 0\} = u(0) = \min_{p'}\{\mathrm{v}(p', p' \cdot 0)\} = \mathrm{v}(p, 0)$$

where the last equality follows from (4).

For the case $m > 0$, fix $(\bar{p}, \bar{m})$ with $\bar{m} > 0$ and define

$$C = \{p \in \boldsymbol{R}^{\mathrm{n}}_{++} : \mathrm{v}(p, \bar{m}) < \mathrm{v}(\bar{p}, \bar{m})\}.$$

Since $\bar{m} > 0$, assumption **P** (monotonicity in p) implies that the set C is nonempty. It also implies that $\bar{p}$ belongs to the boundary of C. Assumption **Q** (quasiconvexity in p) implies that

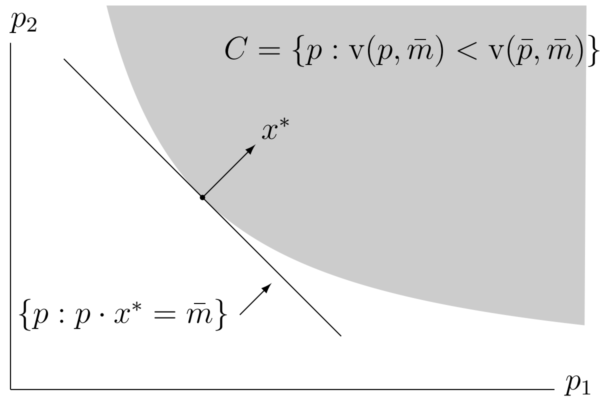

Figure 13.2. x^*

C is convex. Therefore by the Supporting Hyperplane Theorem 10.4.2 there is a nonzero vector x^* properly supporting the closure of C at $\bar p$. (See Figure 13.2.) That is,

$$x^* \cdot \bar p \leqslant x^* \cdot p \text{ for all } p \in C = \{p : \mathrm{v}(p, \bar m) < \mathrm{v}(\bar p, \bar m)\}. \tag{6}$$

Further, **P** implies that C is an increasing set, that is, $p \in C$ and $p' \gg p$ imply that $p' \in C$ so it follows that $x^* \geqq 0$, and we may normalize it so that

$$\bar p \cdot x^* = \bar m. \tag{7}$$

We can strengthen (6) to the following:

$$\mathrm{v}(p, \bar m) < \mathrm{v}(\bar p, \bar m) \quad \Longrightarrow \quad p \cdot x^* > \bar p \cdot x^*. \tag{8}$$

To see this, suppose $\mathrm{v}(p, \bar m) < \mathrm{v}(\bar p, \bar m)$. By upper semicontinuity in m, assumption **S**, for $\lambda > 1$ small enough, $\mathrm{v}(p, \lambda \bar m) < \mathrm{v}(\bar p, \bar m)$. Then setting $\alpha = 1/\lambda < 1$ we have

$$\mathrm{v}(\alpha p, \bar m) = \mathrm{v}(p, \lambda \bar m) < \mathrm{v}(\bar p, \bar m).$$

By (6) we have $x^* \cdot \alpha p \geqslant x^* \cdot \bar p$, but $x^* \geqslant 0$, so $x^* \cdot p > x^* \cdot \alpha p$, and (8) follows. (Note that we do not need to assume upper semicontinuity in p to derive this.)

Now by definition

$$u(x^*) = \inf\{\mathrm{v}(p, p \cdot x^*) : p \in \boldsymbol{R}^{\mathrm{n}}_{++}\}.$$

By **H** (homogeneity of degree zero in p), since x^* is nonzero, we may normalize all prices without changing the infimum, so that

$$u(x^*) = \inf\{\mathrm{v}(p, \bar m) : p \in \boldsymbol{R}^{\mathrm{n}}_{++} \text{ and } p \cdot x^* = \bar m\}. \tag{9}$$

Now by the contrapositive of (8),

$$p \cdot x^* = \bar m \implies \mathrm{v}(p, \bar m) \geqslant \mathrm{v}(\bar p, \bar m).$$

But $\bar p$ satisfies $\bar p \cdot x^* = \bar m$, so

$$\mathrm{v}(\bar p, \bar m) = \min\{\mathrm{v}(p, \bar m) : p \in \boldsymbol{R}^{\mathrm{n}}_{++} \text{ and } p \cdot x^* = \bar m\} = u(x^*).$$

Thus (5) implies

$$\max\{u(x) : \bar p \cdot x \leqslant \bar m\} = u(x^*) = \mathrm{v}(\bar p, \bar m)$$

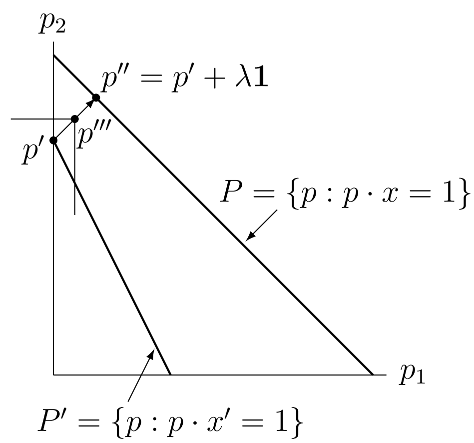

Figure 13.3. p''' is strictly greater than everything near p'.

This completes the proof that v is the indirect utility for u. •

Step 3. u is monotonic:
That is,

$$x' \gg x \implies u(x') > u(x). \tag{10}$$

To see this, let $P = \{p : p \cdot x = 1\}$ and $P' = \{p : p \cdot x' = 1\}$. As we argued above, **P** implies[5]

$$u(x) = \inf\bigl\{\mathrm{v}(p,1) : p \in P\bigr\}, \qquad u(x') = \inf\bigl\{\mathrm{v}(p,1) : p \in P'\bigr\}.$$

Since $x' \gg x$, in particular $x' \gg 0$, so the closure $\overline{P'}$ of P' is compact. Moreover $\overline{P'}$ is disjoint $\overline{P}$ and lies below it. It should be obvious from Figure 13.3 that $u(x) = \inf\{\mathrm{v}(p,1) : p \in P\} < \inf\{\mathrm{v}(p,1) : p \in P'\} = u(x')$, but here is a proof anyhow: There is a sequence $p'_n \in P'$ with $\mathrm{v}(p'_n,1) \to u(x')$. Since $\overline{P'}$ is compact, the sequence must have a subsequence that converges to $p' \in \overline{P'}$. Then there is some $\lambda > 0$ such that the price vector $p'' = p' + \lambda\mathbf{1}$ satisfies $p'' \cdot x = 1$. Let p''' be halfway between p' and p''. Since $p''' \gg p'$ there is some $\varepsilon > 0$ such that if $d(p,p') < \varepsilon$, then $p''' \gg p$. So for n large enough $p''' \gg p'_n$, so $\mathrm{v}(p'_n,1) > \mathrm{v}(p''',1)$. (See Figure 13.3.) Therefore

$$u(x') = \lim_n \mathrm{v}(p'_n,1) \geqslant \mathrm{v}(p''',1) > \mathrm{v}(p'',1) \geqslant \inf_{p \in P} \mathrm{v}(p,1) = u(x).$$

This proves monotonicity. •

Step 4. u is quasiconcave:
That is, for $0 \leqslant \lambda \leqslant 1$,

$$u\bigl(\lambda x + (1-\lambda)x'\bigr) \geqslant \min\bigl\{u(x), u(x')\bigr\}.$$

This is certainly the case if either x or x' is zero, since zero is a global minimizer of u. So assume that neither x nor x' is zero. Then, we have already seen that **H** implies that

$$\begin{aligned} u\bigl(\lambda x + (1-\lambda)x'\bigr) &= \inf\bigl\{\mathrm{v}(p,1) : p \cdot \bigl(\lambda x + (1-\lambda)x'\bigr) = 1\bigr\} \\ &= \inf\bigl\{\mathrm{v}(p,1) : p \cdot \bigl(\lambda x + (1-\lambda)x'\bigr) \leqslant 1\bigr\}, \end{aligned}$$

[5] Note that $x' \gg x$ implies that $x' \neq 0$, so that we may normalize prices by $p \cdot x' = 1$. If $x = 0$, we can't have $p \cdot 0 = 1$, but a simple modification of the following argument shows that $u(x') > u(0)$. I'll leave that to you.

where the second equality follows from **M**. Now if $p \cdot \big(\lambda x + (1-\lambda)x'\big) \leqslant 1$ we must have $p \cdot x \leqslant 1$ or $p \cdot x' \leqslant 1$ (or both.) But $p \cdot x \leqslant 1$ implies $\mathrm{v}(p,1) \geqslant u(x) = \inf\{\mathrm{v}(p,1) : p \cdot x \leqslant 1\}$. Similarly $p \cdot x' \leqslant 1$ implies $\mathrm{v}(p,1) \geqslant u(x')$. Either way,

$$p \cdot \big(\lambda x + (1-\lambda)x'\big) \leqslant 1 \implies \mathrm{v}(p,1) \geqslant \min\big\{u(x), u(x')\big\}.$$

Therefore

$$u\big(\lambda x + (1-\lambda)x'\big) = \inf\Big\{\mathrm{v}(p,1) : p \cdot \big(\lambda x + (1-\lambda)x'\big) \leqslant 1\Big\} \geqslant \min\big\{u(x), u(x')\big\}.$$

So u is quasiconcave. •

Step 5. u is upper semicontinuous:
Recall that u is upper semicontinuous if for each real α the strict lower contour set $\{x : u(x) < \alpha\}$ is open. So fix α and pick $\bar{x}$ such that $u(\bar{x}) < \alpha$. Since

$$u(\bar{x}) = \inf_p \mathrm{v}(p, p \cdot \bar{x}),$$

there is some $\bar{p}$ satisfying

$$\mathrm{v}(\bar{p}, \bar{p} \cdot \bar{x}) < \alpha.$$

By **S** (upper semicontinuity of v in m), there is a neighborhood U of $\bar{x}$ such that $x \in U$ implies

$$\mathrm{v}(\bar{p}, \bar{p} \cdot x) < \alpha.$$

Thus for $x \in U$,

$$u(x) = \inf_p \mathrm{v}(p, p \cdot x) \leqslant \mathrm{v}(\bar{p}, \bar{p} \cdot x) < \alpha.$$

Thus u is upper semicontinuous. •

This completes the proof of the theorem. ∎

References

[1] K. C. Border. 2010. Notes on indirect utilities and the standard of living. http://www.its.caltech.edu/~kcborder/Notes/StandardOfLiving.pdf

[2] W. E. Diewert. 1982. Duality approaches to microeconomic theory. In K. J. Arrow and M. D. Intriligator, eds., *Handbook of Mathematical Economics*, volume 2, chapter 12, pages 535–599. Amsterdam: North Holland.

[3] V. Krishna and H. F. Sonnenschein. 1990. Duality in consumer theory. In J. S. Chipman, D. L. McFadden, and M. K. Richter, eds., *Preferences, Uncertainty, and Optimality: Essays in Honor of Leonid Hurwicz*, chapter 2, pages 44–55. Boulder, Colorado: Westview Press.

[4] K. J. Lancaster. 1958. Welfare propositions in terms of consistency and expanded choice. *Economic Journal* 68(271):464–470. http://www.jstor.org/stable/2227553

[5] A. Mas-Colell, M. D. Whinston, and J. R. Green. 1995. *Microeconomic theory*. Oxford: Oxford University Press.

[6] H. R. Varian. 1992. *Microeconomic analysis*, 3d. ed. New York: W. W. Norton & Co.