Ec 121a
Theory of Value

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Lecture 12: Introduction to Demand Theory

Read Varian [6], Chapters 7, 8, and 9.

The omitted proofs may be found in my on-line notes [2].

12.1 Preference

Preference is comparative notion, so we represent preference as a binary relation on a set X.

$$x \succcurlyeq y \text{ means } x \text{ is as good as } y.$$
$$x \succ y \text{ means } x \text{ is better than } y.$$
$$x \sim y \text{ means } x \text{ and } y \text{ are indifferent.}$$

Formally $\succ$ and $\sim$ may be derived from $\succcurlyeq$ by

$$x \succ y \iff (x \succcurlyeq y \text{ and } \neg\, y \succcurlyeq x)$$
$$x \sim y \iff (x \succcurlyeq y \text{ and } y \succcurlyeq x).$$

Neoclassical economics: Assume that $\succcurlyeq$ is **regular**. That is, it has the following properties.

1. $\succcurlyeq$ is **transitive**: For all x, y, z,

$$(x \succcurlyeq y \;\&\; y \succcurlyeq z) \implies x \succcurlyeq z.$$

2. $\succcurlyeq$ is **complete**: For all x, y,

$$x \succcurlyeq y \text{ or } y \succcurlyeq x \text{ (or both).}$$

For a regular preference $\succcurlyeq$, the strict preference $\succ$ is asymmetric, transitive, and irreflexive. The indifference relation $\sim$ is symmetric, transitive, and reflexive. In other words it is an equivalence relation. The equivalence class $I(x)$ of x, that is,

$$I(x) = \{y \in X : x \sim y\}$$

is called the **indifference curve** through x. These partition the set X. For each $x, y \in X$, we have $x \in I(x)$ and

$$I(x) \cap I(y) \neq \varnothing \implies I(x) = I(y).$$

12.2 Revealed preference

Economists believe that choices reveal preferences: Choosing x when y could have been chosen reveals that $x \succcurlyeq y$. Most of us believe that choice is the only true guide to preference.

12.2.1 Choice functions

- X is a set of alternatives.
- A **budget** is a nonempty subset of X.
- $\mathcal{B}$ is the set of admissible budgets.
- A **choice** (or **choice function**) is a mapping c from $\mathcal{B}$ to subsets of X such that for each budget $B \in \mathcal{B}$,

$$c(B) \subset B.$$

12.2.2 Rational choice

A choice is rational if there are preferences for the choice to reveal. That is, choice c is **rational** if there is some binary relation $\succcurlyeq$ on X such that

$$c(B) = \{x \in B : \text{for all } y \in B,\ x \succcurlyeq y\},$$

in which case we say that $\succcurlyeq$ **rationalizes** c.

[How do we make choosing a set operational?]

12.2.1 Example (A non-rational choice) $X = \{a, b, c\}$, $B_1 = \{a, b, c\}$, $B_2 = \{a, b\}$.

$$c(B_1) = \{a\}, \quad c(B_2) = \{b\}$$

is not rational, as $c(B_1) = \{a\}$ implies $a \succcurlyeq b$ and $a \succcurlyeq a$, so we must have $a \in c(B_2)$ for rationality. □

For more about the consequnces and characterizations of rationality see my on-line notes [1].

12.2.3 Economic well-being (welfare)

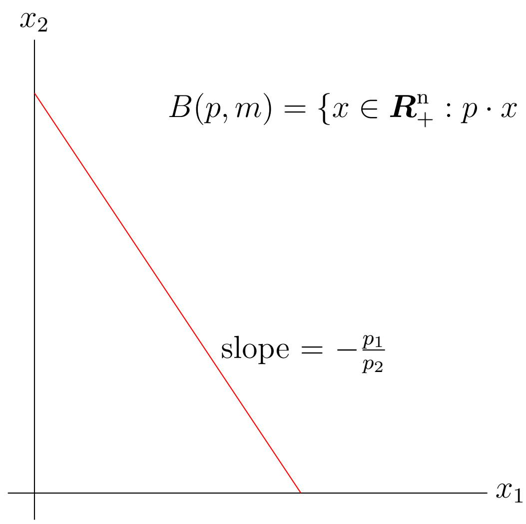
Preferences also reflect economic welfare.

A consumer would be "worse off" if forced to consume something in the budget that is not in the chosen set.

12.3 Prices and budgets

There are n commodities so $X = \boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n : x \geqq 0\}$. That is, alternatives are vectors of quantities of commodities. Think of them as shopping carts if you'd like.

12.3.1 Competitive budgets



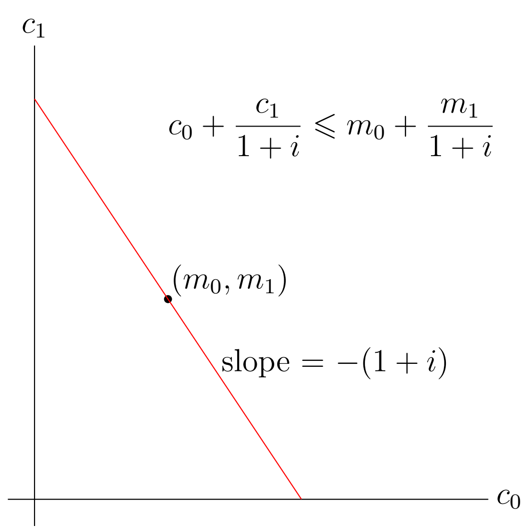
Given price vector $p \gg$ in $\boldsymbol{R}^n$, and income m, the budget is

$$B(p, m) = \{x \in \boldsymbol{R}^n_+ : p \cdot x \leqslant m\}$$

Note, for $\lambda > 0$,

$$B(p, m) = B(\lambda p, \lambda m).$$

12.3.2 Consumption Loans budgets



Two periods, $0, 1$, c_t is consumption at time t, m_t is real income at time t, s is savings (lending) or borrowing at $t = 0$, and i is the interest rate. The temporal budget constraint is

$$\begin{aligned} c_0 &= m_0 - s \\ c_1 &= m_1 + (1+i)s. \end{aligned}$$

Or solving the latter for s and substituting in the former,

$$c_0 + \frac{c_1}{1+i} = m_0 + \frac{m_1}{1+i}.$$

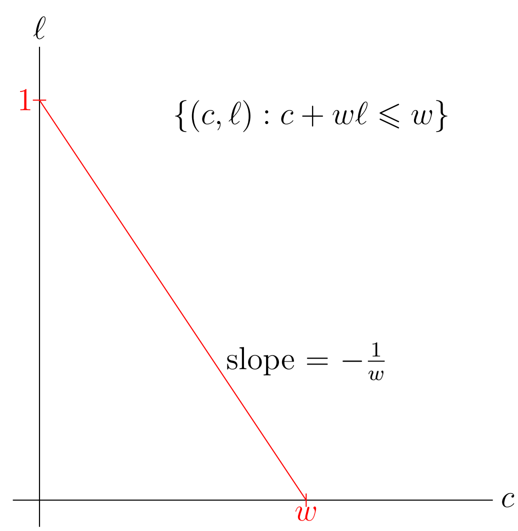
$m_0 + \frac{m_1}{1+i}$ is the **present discounted value** of income. This budget constraint is equivalent to the two separate constraints. For if

$$c_0 + \frac{c_1}{1+i} = m_0 + \frac{m_1}{1+i},$$

define $s = m_0 - c_0$. Then

$$\begin{aligned} c_0 &= m_0 - s \\ c_1 &= m_1 + (1+i)s. \end{aligned}$$

12.3.3 Budget with Labor Income



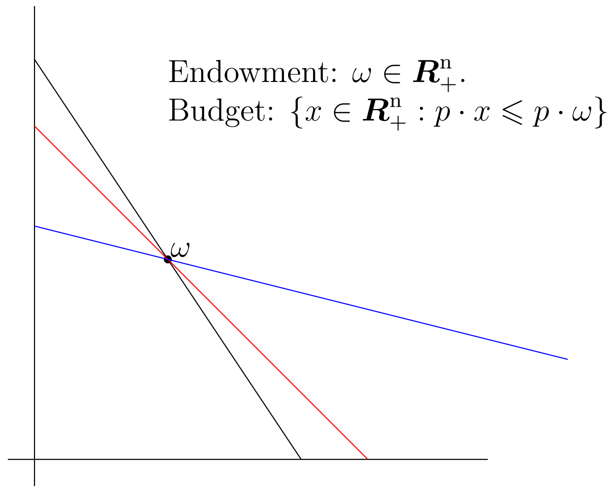
w is the real wage rate. L is labor supplied. Consumption budget is

$$c \leqslant wL.$$

Let 1 be total amount of time in a period. Then $\ell = 1 - L$ is **leisure**. The budget becomes

$$c + w\ell \leqslant w.$$

12.3.4 Pure trade budget



12.4 Normalizing budgets

Budgets are in a sense homogeneous of deree zero. That is,

$$\text{for every } \lambda > 0, \quad B(\lambda p, \lambda m) = B(p, m).$$

Since $(p, m) \in \boldsymbol{R}^{\mathrm{n}+1}$, there are only n degrees of freedom in specifying budget, and we can normmalize a (p, m) by multiplying or dividing by some $\lambda > 0$. Thus we can take as our set of budgets

$$\mathcal{B} = \{B(p, m) : p \gg 0,\ m > 0\},$$

or we could choose some good, say good n, to be the **numéraire**, and set its price to unity. In essence this measures income and prices in terms of units of good n. Since

$$B(p_1, \dots, p_n, m) = B\left(\frac{p_1}{p_n}, \dots, \frac{p_{n-1}}{p_n}, 1, m\right)$$

we can use as or set of budgets

$$\mathcal{B}_n = \{B(p, m) : p \gg 0,\ p_n = 1,\ m > 0\}.$$

Or we could normalize income to unity. Since

$$B(p_1, \dots, p_n, m) = B\left(\frac{p_1}{m}, \dots, \frac{p_n}{m}, 1\right)$$

we can use

$$\mathcal{B}_m = \{B(p, m) : p \gg 0,\ m = 1\}.$$

Or we could divide p by the sum of its components:

$$B(p_1, \dots, p_n, m) = B\left(\frac{p_1}{p_1 + \cdots + p_n}, \dots, \frac{p_n}{p_1 + \cdots + p_n}, \frac{m}{p_1 + \cdots + p_n}\right)$$

and use

$$\mathcal{B}_{\text{sum}} = \{B(p, m) : p \gg 0,\ p_1 + \cdots + p_n = 1,\ m > 0\}.$$

12.5 Preferences over commodity vectors

Preferences may have properties in addition to just regularity.

12.5.1 Definition (Preference sets)

$$P(x) = \{y \in \boldsymbol{R}^n_+ : y \succ x\}, \qquad U(x) = \{y \in \boldsymbol{R}^n_+ : y \succcurlyeq x\},$$
$$P^{-1}(x) = \{y \in \boldsymbol{R}^n_+ : x \succ y\}, \qquad U^{-1}(x) = \{y \in \boldsymbol{R}^n_+ : x \succcurlyeq y\}.$$

12.5.1 Continuity of preferences

12.5.2 Definition *$\succcurlyeq$ is **upper semicontinuous** if for each x, the upper set $U(x)$ is closed.*

*$\succcurlyeq$ is **lower semicontinuous** if for each x, the lower set $U^{-1}(x)$ is closed.*

*$\succcurlyeq$ is **continuous** if it is both upper and lower semicontinuous.*

12.5.3 Proposition *For a regular preference $\succcurlyeq$, the following are equivalent:*

1. *$\succcurlyeq$ is continuous.*
2. *The graph of $\succcurlyeq$, $\{(x, y) \in \boldsymbol{R}^n_+ \times \boldsymbol{R}^n_+ : x \succcurlyeq y\}$, is closed in $\boldsymbol{R}^n_+ \times \boldsymbol{R}^n_+$.*
3. *The graph of $\succ$, $\{(x, y) \in \boldsymbol{R}^n_+ \times \boldsymbol{R}^n_+ : x \succ y\}$, is open in $\boldsymbol{R}^n_+ \times \boldsymbol{R}^n_+$.*

12.5.2 Nonsatiation properties

12.5.4 Definition *$\succcurlyeq$ is **strictly monotonic** if*

$$x \geqq y \,\&\, x \neq y \implies x \succ y.$$

*$\succcurlyeq$ is **monotonic** if*

$$x \gg y \implies x \succ y.$$

*$\succcurlyeq$ is **locally nonsatiated** if for every x and every $\varepsilon > 0$, there exists y satisfying*

$$\|y - x\| < \varepsilon \text{ and } y \succ x.$$

$$\text{strict monotonicity} \implies \text{monotonicity} \implies \text{local nonsatiation}$$

12.5.3 Convexity properties of preferences

12.5.5 Definition *$\succcurlyeq$ is **weakly convex** if*

$$y \succcurlyeq x \implies (1-\lambda)y + \lambda x \succcurlyeq x, \quad 0 < \lambda < 1.$$

*$\succcurlyeq$ is **convex** if*

$$y \succ x \implies (1-\lambda)y + \lambda x \succ x, \quad 0 < \lambda < 1.$$

*$\succcurlyeq$ is **strictly convex** if*

$$y \succcurlyeq x \,\&\, y \neq x \implies (1-\lambda)y + \lambda x \succ x, \quad 0 < \lambda < 1.$$

(Note: convexity does not imply weak convexity unless $\succcurlyeq$ is also upper semicontinuous. See my notes.)

12.5.6 Proposition *Let X be convex, and let $\succcurlyeq$ be a regular preference on X. The following are equivalent.*

1. *$\succcurlyeq$ is weakly convex.*
2. *For each x, the strict upper contour set $P(x)$ is a convex set.*
3. *For each x, the weak upper contour set $U(x)$ is a convex set.*

12.6 Utility

12.6.1 Definition *A function $u\colon X \to \boldsymbol{R}$ is a **utility for** $\succcurlyeq$ if*

$$x \succcurlyeq y \iff u(x) \geqslant u(y).$$

*In this case we say that u **represents** $\succcurlyeq$.*

A utility is never unique. If $f\colon \boldsymbol{R} \to \boldsymbol{R}$ is strictly increasing, then $f \circ u$ is also a utility for $\succcurlyeq$.

Any function $u\colon X \to \boldsymbol{R}$ represents some regular preference on X.

12.6.2 Example (Lexicographic preferences) The **lexicographic order** on the plane is defined by

$$(x_1, x_2) \succcurlyeq (y_1, y_2) \iff (x_1 > y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \geqslant y_2))$$

Note that every indifference "curve" is a singleton!

12.6.3 Fact *There is no utility function that represents the lexicographic order.*

□

12.7 Existence of utility functions

Let $X = \boldsymbol{R}^n_+$ and let $\succcurlyeq$ be a regular preference on X.

12.7.1 Proposition (Debreu [3, pp. 56–59]) *If $\succcurlyeq$ is continuous, then it can be represented by a continuous utility function on X.*

12.7.2 Proposition (Rader [4]–Richter [5]) *If $\succcurlyeq$ is upper semicontinuous, then it can be represented by an upper semicontinuous utility function on X.*

12.7.3 Proposition *If $\succcurlyeq$ is continuous and strictly monotonic, then it can be represented by a strictly increasing continuous utility function on X.*

12.7.4 Proposition *If $\succcurlyeq$ is weakly convex, then any utility is quasiconcave.*

If in addition, $\succcurlyeq$ is convex, then any utility is explicitly quasiconcave.

12.8 Preference Maximization

12.8.1 Weierstrass's Theorem *If B is a nonempty, closed, bounded subset of $\boldsymbol{R}^n$, and $u\colon B \to \boldsymbol{R}$ is continuous, then u has a maximizer in B, that is, there exists $\bar{x} \in$ such that $u(\bar{x}) \geqslant u(x)$ for all $x \in B$.*

12.8.2 Example (Failure of a maximizer to exist) Let $B = [0,1]$ and define $u(x) = \begin{cases} x & x < 1 \\ 0 & x = 1 \end{cases}$. Then no maximizer exists. B is closed and bounded, but u is not continuous.

Let $B = (0,1)$ and define $u(x) = x$. Then no maximizer exists. B is bounded, and u is continuous, but B is not closed.

Let $B = \boldsymbol{R}$ and define $u(x) = x$. Then no maximizer exists. B is closed, and u is continuous, but B is not bounded. □

There is a stronger result.

12.8.3 Definition *A function $f\colon X \to \boldsymbol{R}$ is* ***upper semicontinuous*** *if for every $\alpha \in \boldsymbol{R}$, the set $\{x \in X : f(x) \geqslant \alpha\}$ is closed.*

A function $f\colon X \to \boldsymbol{R}$ is ***lower semicontinuous*** *if for every $\alpha \in \boldsymbol{R}$, the set $\{x \in X : f(x) \leqslant \alpha\}$ is closed.*

12.8.4 Fact *A function $f\colon X \to \boldsymbol{R}$ is continuous if and only if it is both upper and lower semicontinuous.*

12.8.5 Proposition *Let B be a nonempty, closed, bounded subset of $\boldsymbol{R}^n$. If $u\colon B \to \boldsymbol{R}$ is upper semicontinuous, then u has a maximizer in B, that is, there exists $\bar{x} \in B$ such that $u(\bar{x}) \geqslant u(x)$ for all $x \in B$.*

If u is lower semicontinuous then u has a minimizer in B, that is, there exists $\underline{x} \in B$ such that $u(\underline{x}) \leqslant u(x)$ for all $x \in B$.

12.8.6 Definition *The alternative x^* is a $\succcurlyeq$**-greatest** alternative in the set B if $x^* \in B$ and for every $x \in B$, we have $x^* \succcurlyeq x$.*

12.8.7 Proposition *Let $B \subset \boldsymbol{R}^n$ be nonempty, closed and bounded, and assume that the regular preference $\succcurlyeq$ is upper semicontinuous. Then X has a $\succcurlyeq$-greatest element.*

12.8.8 Proposition *Let B be convex, and assume that the regular preference $\succcurlyeq$ is strictly convex. Then a $\succcurlyeq$-greatest element is unique (if it exists).*

12.9 Demand functions

12.9.1 Definition (Demand correspondence)

$$x^*(p,m) = \{x \in B(p,m) : x \text{ is } \succcurlyeq\text{-greatest in } B(p,m)\}.$$

12.9.2 Proposition *If $\succcurlyeq$ is continuous and $p \gg 0$, then $x^*(p,m)$ is nonempty. If $\succcurlyeq$ is strictly convex, then $x^*(p,m)$ is at most a singleton.*

Note that if $p \gg 0$, then $B(p,m)$ is closed and bounded.

12.9.3 Proposition *$x^*(p,m)$ is homogeneous of degree zero in (p,m), that is,*

$$x^*(p,m) = x^*(\lambda p, \lambda m), \qquad \lambda > 0.$$

This is because $B(p,m) = B(\lambda p, \lambda m)$.

12.9.4 Proposition *If $\succcurlyeq$ is locally nonsatiated, then*

$$p \cdot x^*(p,m) = m.$$

12.10 Expenditure minimization and utility maximization

12.10.1 Theorem *If $\succcurlyeq$ is a continuous and locally nonsatiated regular preference on $\boldsymbol{R}^n_+$, and if $p \gg 0$ and $m > 0$, then*

$$x^* \text{ maximizes } \succcurlyeq \text{ over } B(p,m) \iff x^* \text{ minimizes } p \cdot x \text{ over } U(x^*).$$

12.10.2 Example (Why $m > 0$ and/or $p \gg 0$ is needed) Let $X = \boldsymbol{R}^2_+$. Let preferences be defined by the utility function $u(x_1, x_2) = x_1 + x_2$. (This preference relation is continuous, convex, and locally nonsatiated.) Let $x^* = (1,0)$ and $p = (0,1)$. Then x^* minimizes $p \cdot x$ over $U(x^*)$. But $B(p, p \cdot x^*) = B(p, 0)$, which is just the x_1-axis. This budget set is unbounded and no $\succcurlyeq$-greatest element exists. See Figure 12.1. □

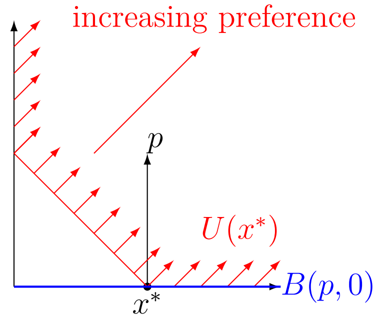


Figure 12.1. Example of nonequivalence of expenditure minimization and utility maximization.

12.11 Indirect utility and expenditure functions

Let x^* denote the ordinary demand function. The **indirect utility** v is the optimal value function for the consumer's utility maximization problem.

$$\mathrm{v}(p,m) = u\big(x^*(p,m)\big).$$

12.11.1 Proposition v *is quasi-convex in* (p,m)*, homogeneous of degree zero in* (p,m)*.*

Proof: Homogeneity follows from $B(\lambda p, \lambda m) = B(p,m)$. For quasiconvexity, we need to show that for any (p,m) and (p',m'), and $0 \leqslant \lambda \leqslant 1$ that

$$\mathrm{v}\big((1-\lambda)p + \lambda p', (1-\lambda)m + \lambda m'\big) \leqslant \max\{\mathrm{v}(p,m), \mathrm{v}(p',m')\}.$$

So let x_λ be demanded from the budget $B_\lambda = B\big((1-\lambda)p + \lambda p', (1-\lambda)m + \lambda m'\big)$. Observe that x_λ must belong to at least one of $B(p,m)$ or $B(p',m')$. For if this were not the case, we would have $p \cdot x_\lambda > m$ and $p' \cdot x_\lambda > m'$, which would imply that $\big((1-\lambda)p + \lambda p'\big) \cdot x_\lambda > (1-\lambda)m + \lambda m'$, contradicting the the fact that x_λ was chosen from B_λ.

Now if $x_\lambda \in B(p,m)$, by definition of the indeirect utility v we would have $u(x_\lambda) \leqslant \mathrm{v}(p,m)$. Ditto for (p',m'), so

$$\mathrm{v}\big((1-\lambda)p + \lambda p', (1-\lambda)m + \lambda m'\big) = u(x_\lambda) \leqslant \max\{\mathrm{v}(p,m), \mathrm{v}(p',m')\}.$$

∎

The **expenditure function** e is the optimal value function for the expenditure minimization problem

$$\underset{x}{\text{minimize}}\, p \cdot x \text{ subject to } u(x) \geqslant v$$

The solution $\hat{x}(p,v)$ is called the **Hicksian compensated demand**.

12.11.2 Proposition $\hat{x}$ *is homogeneous of degree 1 in* p*.*

The expenditure function is

$$e(p,v) = p \cdot \hat{x}(p,v)$$

References

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