Ec 121a KC Border
Theory of Value Fall 2020

Lecture 10: Lagrange Multipliers and Decentralization

10.1 Convexity of PPS and Support Points on the PPF

Recall that the **production possibility set** (PPS) is

Should I use subscripts or superscripts on the production functions and the input vectors?

$$\left\{ y \in \boldsymbol{R}^{\mathrm{n}} : 0 \leqslant y^j \leqslant f^j(v^j),\ v^j \geqq 0,\ j = 1, \dots, n, \text{ and } \sum_{j=1}^{n} v^j \leqq \omega \right\}.$$

Note that the PPS is compact, since the f^js are assumed to be continuous and monotonic, so the PPS is the continuous image of the compact set

$$\left\{ (v^1, \dots, v^n) \in \boldsymbol{R}^{\ell \mathrm{n}} : v^j \geqq 0,\ j = 1, \dots, n, \text{ and } \sum_{j=1}^{n} v^j \leqq \omega \right\}.$$

The **production possibility frontier** (PPF) is the outer boundary of the PPS. That is, y belongs to the PPF if y belongs to the PPS and there is no y' in the PPS distinct from y with $y' \geqq y$. Such a y is also called **technically efficient**.

It is easy to verify that if each f^j is concave, then the PPS is convex. Therefore every point on the PPF is a support point. That is, if y belongs to the PPF, then there is a vector p of strictly positive prices such that y maximizes p over the PPS. (See the appendix.)

Prove the convexity.

This follows from the separating hyperplane theorem applied to the PPS and $\{z : z \gg y\}$. In this case the PPF can be parametrized by p.

10.2 Decentralization and Lagrange Multipliers

Maximizing the value of output:

$$\begin{aligned} \underset{v^1,\dots,v^n}{\text{maximize}} \sum_{j=1}^{n} p_j f^j(v^j) \quad & \text{subject to} \\ \sum_{j=1}^{n} v_k^j \leqslant \omega_k \quad & k = 1, \dots, \ell \\ v_k^j \geqslant 0 \quad & j = 1, \dots, n \\ & k = 1, \dots, \ell. \end{aligned}$$

The Lagrangean is:

$$L(v, \mu) = \sum_{j=1}^{n} p_j f^j(v^j) + \sum_{k=1}^{\ell} \mu_k \left(\omega_k - \sum_{j=1}^{j} v_k^j \right).$$

Note that as long as each $\omega_k > 0$, then Slater's Condition is satisfied. So a point v^* solves the maximization problem if and only if the there are λ^* and μ^* such that $v^*; \lambda^*, \mu^*)$ is a saddlepoint of the Lagrangean.

Let's examine a simplified version with $n = 2$ and $\ell = 2$, and let's further name the inputs labor, L, and capital, K, available in fixed quantities $\bar{L}$ and $\bar{K}$, and let us also use w and r for the Lagrange multipliers instead of μ. (The same argument works in the general case—there is just more notation.) The saddlepoint condition is

$$\begin{aligned} &p_1 f^1(L_1, K_1) + p_2 f^2(L_2, K_2) + w^*(\bar{L} - L_1 - L_2) + r^*(\bar{K} - K_1 - K_2) \\ &\quad \leqslant p_1 f^1(L_1^*, K_1^*) + p_2 f^2(L_2^*, K_2^*) + w^*(\bar{L} - L_1^* - L_2^*) + r^*(\bar{K} - K_1^* - K_2^*) \end{aligned} \tag{1}$$

$$\begin{aligned} &p_1 f^1(L_1^*, K_1^*) + p_2 f^2(L_2^*, K_2^*) + w^*(\bar{L} - L_1^* - L_2^*) + r^*(\bar{K} - K_1^* - K_2^*) \\ &\quad \leqslant p_1 f^1(L_1^*, K_1^*) + p_2 f^2(L_2^*, K_2^*) + w(\bar{L} - L_1^* - L_2^*) + r(\bar{K} - K_1^* - K_2^*) \end{aligned} \tag{2}$$

where (1) holds for all $(L_1, L_2, K_1, K_2) \geqq 0$ and (2) holds for all $(\lambda, w, r) \geqq 0$. Evaluating (1) at $L_1 = L_1^*$ and $K_1 = K_1^*$, and canceling common terms yields

$$p_2 f^2(L_2, K_2) - w^* L_2 - r^* K_2 \leqslant p_2 f^2(L_2^*, K_2^*) - w^* L_2^* - r^* K_2^*$$

This says that (L_2^*, K_2^*) maximizes profit at price p_2 and wages w^* and rental rate r^*.

Similarly, evaluating (1) at $L_2 = L_2^*$ and $K_2 = K_2^*$, and canceling common terms yields

$$p_1 f^1(L_1, K_1) - w^* - L_1 - r^* K_1 \leqslant p_1 f^1(L_1^*, K_1^*) - w^* L_1^* + -^* K_1^*$$

which says that (L_1^*, K_1^*) maximizes profit at price p_1 and wages w^* and rental rate r^*.

In other words,

> *the Lagrange multipliers on the resource constraints are prices that decentralize the problem of maximizing the value of output.*

But the Saddlepoint Theorem also tells us we can go backwards! That is, if we maximize profits given wages and the resource markets clear, then profit maximization leads to maximization of output value.

That is, suppose (L_i^*, K_i^*) maximizes $p_i f^i(L, K) - wL - rK$, for $i = 1, 2$, and assume that $K_1^* + K_2^* = \bar{K}$ and $L_1^* + L_2^* = \bar{L}$. Then we have

$$\begin{aligned} &p_1 f^1(L_1, K_1) - wL_1 - rK_1 + p_2 f^2(L_2, K_2) - wL_2 - rK_2 \\ &\qquad \leqslant p_1 f^1(L_1^*, K_1^*) - wL_1^* - rK_1^* + p_2 f^2(L_2^*, K_2^*) - wL_2^* - rK_2^* \end{aligned}$$

for all L_1, K_1, L_2, K_2. Add $w\bar{L} + r\bar{K}$ to both sides and rearrange to get

$$\begin{aligned} p_1 f^1(L_1, K_1) + p_2 f^2(L_2, K_2) + w(\bar{L} - L_1 - L_2) + r(\bar{K} - K_1 - K_2) \\ \leqslant p_1 f^1(L_1^*, K_1^*) + p_2 f^2(L_2^*, K_2^*) + w(\bar{L} - L_1^* - L_2^*) + r(\bar{K} - K_1^* - K_2^*) \end{aligned}$$

for all L_1, K_1, L_2, K_2. This is the first saddlepoint inequality. The second saddlepoint inequality is

$$\begin{aligned} p_1 f^1(L_1^*, K_1^*) + p_2 f^2(L_2^*, K_2^*) + w(\bar{L} - L_1^* - L_2^*) + r(\bar{K} - K_1^* - K_2^*) \\ \leqslant p_1 f^1(L_1^*, K_1^*) + p_2 f^2(L_2^*, K_2^*) + w'(\bar{L} - L_1^* - L_2^*) + r'(\bar{K} - K_1^* - K_2^*) \end{aligned}$$

for all $(w', r') \geqq (0, 0)$, which is true since the resource constraints are assumed to bind.

We could also have used the Arrow–Enthoven Theorem and the first order conditions to obtain the same result.

10.3 Factor Price Equalization[1]

Two factors of production, call them labor L and capital K. Two outputs y_1 and y_2 with CRS production functions f^1 and f^2. To maximize the value of output at prices p_1 and p_2, consider the isoquants for a dollar's worth of output:

$$\left\{x : f^1(x) = \frac{1}{p_1}\right\} \quad \text{and} \quad \left\{x : f^2(x) = \frac{1}{p_2}\right\}$$

The convex hull of the upper contour sets gives the input requirement set for a dollar's worth of output. See Figure 10.1.

The way the output is determined is this: Draw a ray from the origin to the aggregate $(\bar{L}, \bar{K})$, and find the point where it crosses the iso-dollar curve. If this point lies on the $f^1 = 1/p_1$ curve, then only good 1 is produced. If it lies on the $f^2 = 1/p_2$ curve, then only good 2 is produced. But if it lies on the line segment, then the country diversifies and produce both. The set of $(\bar{L}, \bar{K})$ for which this occurs is called the **d**iversification cone.

> If the aggregate endowment lies in the diversification cone then the wage/rental ratio is just the slope of the straight line segment.

10.3.1 Example We can compute this slope for the Cobb–Douglas case.

$$y_i = f^i(L, K) = L^{\alpha_i} K^{1-\alpha_i}$$

The isoquants are given by

$$\hat{K}^i(L) = p_i^{-\frac{1}{1-\alpha_i}} L^{-\frac{\alpha_i}{1-\alpha_i}}$$

[1] This section is based on Chipman [1].

$f^2(L,K) = \frac{1}{p_2}$

K

diversification cone

common tangent

```
Plot[{K1[x], K2[x], line[x], K1[L1] x / L1, K2[L2] x / L2},
 {x, 0, 2 L1}, PlotRange → {0, 2 K2[L2]}, AspectRatio → Automatic,
 PlotStyle → {{Cyan}, {Blue}, {Green}, {Black}, {Black}}, Ticks → None]
```

$f^1(L,K) = \frac{1}{p_1}$

L

K

```
Show[Plot[K1[x], {x, L1, 2 L1}, PlotRange → {0, 2 K2[L2]}, AspectRatio → Automatic,
  PlotStyle → {{Cyan}}], Plot[K2[x], {x, 0, L2}, PlotRange → {0, 2 K2[L2]},
  AspectRatio → Automatic, PlotStyle → {{Blue}}], Plot[line[x], {x, L1, L2},
  PlotRange → {0, 2 K2[L2]}, AspectRatio → Automatic, PlotStyle → {{Green}}],
 Plot[{K1[L1] x / L1, K2[L2] x / L2}, {x, 0, 2 L1}, PlotRange → {0, 2 K2[L2]}, AspectRatio → Automatic,
  PlotStyle → {{Black}, {Black}}], PlotRange → All, AxesOrigin → {0, 0}, Ticks → None]
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L

Figure 10.1. Isoquant for a dollar of output.

The slope of the isoquant is then

$$\frac{d\hat{K}_i}{dL_i} = -\frac{\alpha_i}{1-\alpha_i} p_i^{-\frac{1}{1-\alpha_i}} L_i^{-\frac{1}{1-\alpha_i}},$$

where K is on the vertical axis and L is on the horizontal axis. To find the common tangent line we need to find $\tilde{L}_i$ and $\tilde{K}_i = \hat{K}_i(\tilde{L}_i)$ satisfying

$$\frac{d\hat{K}_1(\tilde{L}_1)}{dL} = \frac{d\hat{K}_2(\tilde{L}_2)}{dL} = m$$

and

$$m = -\frac{\tilde{K}_2 - \tilde{K}_1}{\tilde{L}_2 - \tilde{L}_1}.$$

The solution to this is

You should check this for typos.

$$\begin{aligned}
\tilde{L}_1 &= \left[\frac{1-\alpha_2}{1-\alpha_1} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{\alpha_2}{1-\alpha_2}} p_1^{-\frac{1}{1-\alpha_1}} p_2^{\frac{1}{1-\alpha_2}} \right]^{\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_1-\alpha_2}} \\
&= \alpha_1^{\frac{\alpha_2(1-\alpha_1)}{\alpha_2-\alpha_1}} \alpha_2^{\frac{\alpha_2(1-\alpha_1)}{\alpha_1-\alpha_2}} (1-\alpha_1)^{\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_2-\alpha_1}} (1-\alpha_2)^{\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_1-\alpha_2}} p_1^{\frac{1-\alpha_2}{\alpha_2-\alpha_1}} p_2^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}} \\
\tilde{K}_1 &= \alpha_1^{\frac{\alpha_1\alpha_2}{\alpha_1-\alpha_2}} \alpha_2^{\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}} (1-\alpha_1)^{\frac{\alpha_1(1-\alpha_2)}{\alpha_1-\alpha_2}} (1-\alpha_2)^{\frac{\alpha_1(1-\alpha_2)}{\alpha_2-\alpha_1}} p_1^{\frac{\alpha_2}{\alpha_1-\alpha_2}} p_2^{\frac{\alpha_1}{\alpha_2-\alpha_1}} \\
\tilde{L}_2 &= \left[\frac{1-\alpha_1}{1-\alpha_2} \left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{\alpha_1}{1-\alpha_1}} p_1^{\frac{1}{1-\alpha_1}} p_2^{-\frac{1}{1-\alpha_2}} \right]^{\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_2-\alpha_1}} \\
&\text{etc.}
\end{aligned}$$

(Note that $\alpha_1 \tilde{L}_2 = \alpha_2 \tilde{L}_1$.) Substituting into either of the expressions for the slope we get

$$m = -(1-\alpha_1)^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}} (1-\alpha_2)^{\frac{1-\alpha_2}{\alpha_2-\alpha_1}} \alpha_1^{\frac{\alpha_1}{\alpha_1-\alpha_2}} \alpha_2^{\frac{\alpha_2}{\alpha_2-\alpha_1}} p_1^{\frac{1-\alpha_2}{\alpha_2-\alpha_1}} p_2^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}}$$

Thus the wage/rental ratio satisfies

$$\frac{w}{r} = -m.$$

We also know that in the Cobb–Douglas CRS case α_i is labor's share and $1-\alpha_i$ is capital's share, so

$$w = \frac{\alpha_i}{\tilde{L}_i}, \qquad r = \frac{1-\alpha_i}{\tilde{K}_i},$$

or

$$w = \alpha_1^{\frac{\alpha_1(1-\alpha_2)}{\alpha_1-\alpha_2}} \alpha_2^{\frac{\alpha_2(1-\alpha_1)}{\alpha_2-\alpha_1}} (1-\alpha_1)^{\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_1-\alpha_2}} (1-\alpha_2)^{\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_2-\alpha_1}} p_1^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}} p_2^{\frac{1-\alpha_1}{\alpha_2-\alpha_1}}$$

and

$$r = \alpha_1^{\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}} \alpha_2^{\frac{\alpha_1\alpha_2}{\alpha_1-\alpha_2}} (1-\alpha_1)^{\frac{\alpha_2(1-\alpha_1)}{\alpha_2-\alpha_1}} (1-\alpha_2)^{\frac{\alpha_1(1-\alpha_2)}{\alpha_1-\alpha_2}} p_1^{\frac{\alpha_2}{\alpha_2-\alpha_1}} p_2^{\frac{\alpha_1}{\alpha_1-\alpha_2}}$$

□

Leave it as an exercise to compute w and r outside the diversification cone.

Also, the Cobb–Douglas case satisfies the **absence of factor intensity reversal** condition, which is needed to guarantee that countries with identical technology, immobile factors, but common output prices will have identical factor wages if they both diversify.

10.4 Appendix: Supporting hyperplanes

10.4.1 Definition *Let C be a set in $\boldsymbol{R}^{\mathrm{m}}$ and x a point belonging to C. The nonzero real-valued linear function p* ***supports C at x*** *(from below) if $p \cdot y \geqslant p \cdot x$ for all $y \in C$, and we may write $p \cdot C \geqslant p \cdot x$. The hyperplane $\{y : p \cdot y = p \cdot x\}$ is a* ***supporting hyperplane for C at x****. The support is* ***proper*** *if $p \cdot y \neq p \cdot x$ for some y in C. We may also say that the half-space $\{z : p \cdot z \geqslant p \cdot x\}$ supports C at x if p supports C at from below, etc.*

10.4.2 Finite Dimensional Supporting Hyperplane Theorem *Let C be a convex subset of $\boldsymbol{R}^{\mathrm{n}}$ and let $\bar{x}$ belong to C. Then there is a hyperplane properly supporting C at $\bar{x}$ if and only if $\bar{x} \notin \operatorname{ri} C$, where $\operatorname{ri} C$ is the interior of C relative to the smallest affine subspace that includes it.*

See my on-line notes.

References

[1] J. S. Chipman. 1966. A survey of the theory of international trade: Part 3, the modern theory. *Econometrica* 34(1):18–76.
http://www.jstor.org/stable/1909855