Lecture 8: Production Functions, Cost Functions, and Production Possibilities

8.1 A Simple Model of Production Possibilities

This is a very simple model of the production possibilities of an economy. The framework is based on A. P. Lerner [2].

There are n outputs $y_1, \ldots, y_n$ and ℓ factors $v_1, \ldots, v_\ell$. Each output is produced according to the production function $y_j = f^j(v_1^j, \ldots, v_\ell^j)$. There is no joint production, there are no intermediate goods, and there is only one production function for each output.

The supply of factors in the economy are fixed at levels $\omega_1, \ldots, \omega_\ell$.

Assume that for each j, the production function satisfies

$$f^j \colon \boldsymbol{R}^\ell_+ \to \boldsymbol{R} \text{ is continuous, } C^2 \text{ on } \boldsymbol{R}^\ell_{++},\ \nabla f^j \gg 0 \text{ on } \boldsymbol{R}^\ell_{++},$$

and that the Hessian

$$D^2 f^j \text{ is negative definite on the subspace orthogonal to } \nabla f^j.$$

You will presently see why we make these assumptions. They guarantee that all the second order conditions hold as strict inequalities.

8.1.1 Production possibility frontier

The **production possibility set (PPS)** is

$$\Big\{ y \in \boldsymbol{R}^{\mathrm{n}} : 0 \leqslant y^j \leqslant f^j(v^j),\ v^j \geqq 0,\ j = 1, \ldots, n, \text{ and } \sum_{j=1}^n v^j \leqq \omega \Big\}.$$

Note that the PPS is compact, since the f^js are continuous and monotonic, so the PPS is the continuous image of the compact set

$$\Big\{ (v^1, \ldots, v^n) \in \boldsymbol{R}^{\ell \mathrm{n}} : v^j \geqq 0,\ j = 1, \ldots, n, \text{ and } \sum_{j=1}^n v^j \leqq \omega \Big\}.$$

The **production possibility frontier** (PPF) is the outer boundary of the PPS.

The PPF solves a constrained maximization problem

The production possibility frontier can be characterized by the following maximization problem.

$$\underset{v^1,\dots,v^n}{\text{maximize}}\, f^n(v^n) \quad \text{subject to}$$

$$\begin{aligned} f^j(v^j) &= \eta_j, & j &= 1,\dots,n-1 \\ \sum_{j=1}^n v^j_k, &= \omega_k & k &= 1,\dots,\ell \\ v^j_k &\geqq 0, & j &= 1,\dots,n \\ && k &= 1,\dots,\ell. \end{aligned}$$

The Lagrangean is:

$$L(v,\lambda,\mu;\eta,\omega) = \\ f^n(v^n_1,\dots,v^n_\ell) + \sum_{j=1}^{n-1} \lambda_j \Big(f^j(v^j_1,\dots,v^j_\ell) - \eta_j \Big) + \sum_{k=1}^{\ell} \mu_k \Big(\omega_k - \sum_{j=1}^n v^j_k \Big).$$

In order to apply the LMT we need to verify that the Lagrange Constraint Qualification is satisfied. That is, we need to show that the gradients of the constraints are linearly independent (at the optimum). To see this, it might help to consult Table 8.1. Suppose $\lambda_1,\dots,\lambda_{n-1},\mu_1,\dots,\mu_\ell$ are coefficients on the gradients that yield a linear combination of the gradients that adds up to the zero vector. Then the last ℓ rows of Table 8.1 imply that $\mu_1 = \cdots = \mu_\ell = 0$. Thus since each $f^j_k > 0$, we get $\lambda_j = 0$, for all $j = 1,\dots,n-1$. That is, the gradients are linearly indepenednt.

Thus by the Lagrange multiplier theorem, there are Lagrange multipliers $\hat\lambda_j$, $\hat\mu_k$, such that the first order conditions are (assuming each $\hat v^j_k > 0$) :

$$\hat\lambda_j f^j_k(\hat v^j) - \hat\mu_k = 0 \qquad \begin{aligned} j &= 1,\dots,n \\ k &= 1,\dots,\ell \end{aligned}$$

where for symmetry we define $\hat\lambda_n = 1$. This implies

$$\hat\lambda_j = \frac{f^n_k}{f^j_k}$$

for any input $k = 1,\dots,\ell$.

Let $\hat y_n(\eta,\omega)$ be the optimal value function. Its graph *is* the PPF. By the envelope theorem, the slope of the PPF satisfies

$$\boxed{\frac{\partial \hat y_n}{\partial \eta_j} = \frac{\partial L}{\partial \eta_j} = -\hat\lambda_j = -\frac{f^n_k}{f^j_k}}$$

	λ_1			λ_j			λ_{n-1}	μ_1	μ_ℓ
v_1^1	f_1^1	0					0	−1	0
v_ℓ^1	f_ℓ^1	0					0	0	−1
$\vdots$									
v_1^j	0		0	f_1^j	0		0	−1	0
v_ℓ^j	0		0	f_ℓ^j	0		0	0	−1
$\vdots$									
v_1^{n-1}	0					0	f_1^{n-1}	−1	0
v_ℓ^{n-1}	0					0	f_ℓ^{n-1}	0	−1
v_1^n	0						0	−1	0
v_ℓ^n	0						0	0	−1

Table 8.1. Columns are the gradients of constraints, indexed by the Lagrange multipliers in the top row. Rows list the partial derivative with respect to the variable in the far left column.

for any $j = 1, \dots, n-1$, $k = 1, \dots, \ell$. In other words, λ_j is the marginal opportunity cost of a unit of y_j in terms of y_n.

Also note that

$$\frac{f^j_k}{f^j_{k'}} = \frac{\hat{\mu}_k}{\hat{\mu}_{k'}},$$

which is independent of j. That is, in every industry the slopes of the isoquants are the same.

Second order conditions

While we're at it, let's check the second order conditions. The Hessian of the Lagrangean is the block diagonal $\ell n \times \ell n$ matrix

$$H = \begin{bmatrix} \lambda_1 f^1_{11} & \lambda_1 f^1_{1\ell} & & & & \\ & & & & & \\ \lambda_1 f^1_{\ell 1} & \lambda_1 f^1_{\ell\ell} & & & & \\ & & \lambda_2 f^2_{11} & \lambda_2 f^2_{1\ell} & & \\ & & & & & \ddots \\ & & \lambda_2 f^2_{\ell 1} & \lambda_2 f^2_{\ell\ell} & & \\ & & & & \lambda_n f^n_{11} & \lambda_n f^n_{1\ell} \\ & & & & & \\ & & & & \lambda_n f^n_{\ell 1} & \lambda_n f^n_{\ell\ell} \end{bmatrix}$$

Let $x = (x^1, \dots, x^n)$ belong to $\boldsymbol{R}^{\ell n}$. The second order condition is that the quadratic form $x'Hx$ is negative semidefinite on the subspace orthogonal to the gradients of the constraints. Again referring to Table 8.1, it is straightforward to see that this means

$$\sum_{j=1}^{n} \left[\sum_{i=1}^{\ell} \sum_{k=1}^{\ell} \lambda_j f^j_{ik} x^j_i x^j_k \right] \leqslant 0$$

for all nonzero x satisfying

$$\nabla f^j \cdot x^j = \sum_{k=1}^{\ell} f^j_k x^j_k = 0, \qquad j = 1, \dots, n-1,$$

and

$$\sum_{j=1}^{n} x^j_k = 0 \qquad k = 1, \dots, \ell.$$

What about the case $j = n$? If we can show that $\nabla f^n \cdot x^n = 0$, then by our assumption on the gradients of the f^js, each $\lambda_j > 0$, so by the assumption on the Hessian of the f^js, each bracketed term is nonpositive, and at least one is strictly negative (since at least one $x^j \neq 0$).

To see that $\nabla f^n \cdot x^n = 0$, observe that for each k, $x^j_k = -\sum_{j=1}^n x^j_k$. Thus

$$\begin{aligned}\nabla f^n \cdot x^n &= \sum_{k=1}^{\ell} f^n_k x^n_k \\ &= -\sum_{k=1}^{\ell} f^n_k \sum_{j=1}^{n-1} x^j_k \\ &= -\sum_{j=1}^{n-1} \left[\sum_{k=1}^{\ell} \lambda_j f^j_k x^j_k\right] \\ &= 0.\end{aligned}$$

The penultimate equality follows from the first order condition that $\lambda_j f^j_k = \mu_k = f^n_k$ for all i.

8.1.2 Relation to cost minimization

Assume that each producer faces the same wages $w = (w_1, \dots, w_\ell)$ for the factors and minimizes costs. To ease notation in this section, I shall suppress the superscripts denoting the particular output.

The cost minimization problem is to

$$\text{minimize } w \cdot v \quad \text{subject to } f(v) \geqslant y.$$

Form the Lagrangean

$$L(v, \gamma; w, y) = w \cdot v - \gamma\Big(f(v) - y\Big).$$

The value function is the cost function $c(w, y)$. By the envelope theorem, the marginal cost is

$$\text{MC} = \frac{\partial c}{\partial y} = \frac{\partial L}{\partial y} = \gamma.$$

We also have the first order conditions (check the gradient of the constraint) :

$$w_k - \gamma f_k = 0, \qquad k = 1, \dots, \ell$$

assuming each $v_k > 0$. (Note that these implies $\gamma > 0$.) In other words,

$$f_k = \frac{w_k}{\text{MC}}$$

Now back to the PPF. If all firms face the same wages and minimize costs, then

$$\frac{\partial \hat{y}_n}{\partial \eta_j} = -\hat{\lambda}_j = -\frac{f^n_k}{f^j_k} = -\frac{\frac{w_k}{\text{MC}_n}}{\frac{w_k}{\text{MC}_j}} = -\frac{MC_j}{MC_n}.$$

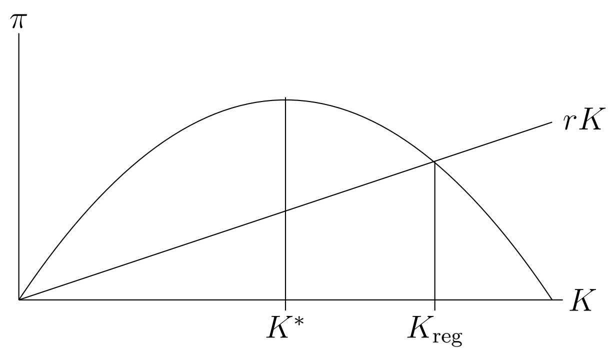
That is, the marginal opportunity cost of one unit of y_j expressed in terms of y_n is exactly the ratio of the marginal cost of a unit of y_j (calculated in terms of wages) relative to the marginal cost of a unit of y_n. What this tells us is that marginal costs (derived from wages) indicate real opportunity costs!

8.2 The Averch–Johnson Effect

Averch and Johnson [1] pointed out that a firm subject to rate of return regulation has an incentive not to minimize costs. Thus the apparent cost function for these firms does not yield the true production function. Rate of return regulation is baesed on a couple of Supreme Court rulings:

- *Munn v. Illinois*
- *Hope* case

Maximize $\pi(K)$ subject to $\pi \leqslant rK$. If the constraint binds the picture looks like this:



The regulated firm overuses capital in order to get a higher rate base.

8.3 Quasiconcave functions

There are weaker notions of convexity that are commonly applied in economic theory.

8.3.1 Definition *A function $f\colon C \to \boldsymbol{R}$ on a convex subset C of a vector space is:*

- ***quasiconcave*** *if for all x, y in C with $x \neq y$ and all $0 < \lambda < 1$*

$$f\Big(\lambda x + (1-\lambda)y\Big) \geqslant \min\{f(x), f(y)\}.$$

- ***strictly quasiconcave*** *if for all x, y in C with $x \neq y$ and all $0 < \lambda < 1$*

$$f\Big(\lambda x + (1-\lambda)y\Big) > \min\{f(x), f(y)\}.$$

• ***explicitly quasiconcave*** *or* ***semistrictly quasiconcave*** *if it is quasiconcave and in addition, for all x, y in C with $x \neq y$ and all $0 < \lambda < 1$*

$$f(x) > f(y) \implies f\Big(\lambda x + (1-\lambda)y\Big) > \min\{f(x), f(y)\} = f(y).$$

• ***quasiconvex*** *if for all x, y in C with $x \neq y$ and all $0 < \lambda < 1$*

$$f\Big(\lambda x + (1-\lambda)y\Big) \leqslant \max\{f(x), f(y)\}.$$

• ***strictly quasiconvex*** *if for all x, y in C with $x \neq y$ and all $0 < \lambda < 1$*

$$f\Big(\lambda x + (1-\lambda)y\Big) < \max\{f(x), f(y)\}.$$

• ***explicitly quasiconvex*** *or* ***semistrictly quasiconvex*** *if it is quasiconvex and in addition, for all x, y in C with $x \neq y$ and all $0 < \lambda < 1$*

$$f(x) < f(y) \implies f\Big(\lambda x + (1-\lambda)y\Big) < \max\{f(x), f(y)\} = f(y).$$

There are other choices we could have made for the definition based on the next lemma.

8.3.2 Lemma *For a function $f\colon C \to \boldsymbol{R}$ on a convex set, the following are equivalent:*

1. *The function f is quasiconcave.*

2. *For each $\alpha \in \boldsymbol{R}$, the strict upper contour set $[f(x) > \alpha]$ is convex, but possibly empty.*

3. *For each $\alpha \in \boldsymbol{R}$, the upper contour set $[f(x) \geqslant \alpha]$ is convex, but possibly empty.*

Proof: (1) $\implies$ (2) If f is quasiconcave and x, y in C satisfy $f(x) > \alpha$ and $f(y) > \alpha$, then for each $0 \leqslant \lambda \leqslant 1$ we have

$$f\Big(\lambda x + (1-\lambda)y\Big) \geqslant \min\{f(x), f(y)\} > \alpha.$$

(2) $\implies$ (3) Note that

$$[f \geqslant \alpha] = \bigcap_{n=1}^{\infty} [f > \alpha - \tfrac{1}{n}],$$

and recall that the intersection of convex sets is convex.

(3) $\implies$ (1) If $[f \geqslant \alpha]$ is convex for each $\alpha \in \boldsymbol{R}$, then for $y, z \in C$ put $\alpha = \min\{f(y), f(z)\}$ and note that $f\Big(\lambda y + (1-\lambda)z\Big)$ belongs to $[f \geqslant \alpha]$ for each $0 \leqslant \lambda \leqslant 1$. ∎

8.3.3 Corollary *A concave function is quasiconcave. A convex function is quasiconvex.*

8.3.4 Lemma *A strictly quasiconcave function is also explicitly quasiconcave. Likewise a strictly quasiconvex function is also explicitly quasiconvex.*

Of course, not every quasiconcave function is concave.

8.3.5 Example (Explicit quasiconcavity) This example sheds some light on the definition of explicit quasiconcavity. Define $f\colon \boldsymbol{R} \to [0,1]$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$$

If $f(x) > f(y)$, then $f\Big(\lambda x + (1-\lambda)y\Big) > f(y)$ for every $\lambda \in (0,1)$ (since $f(x) > f(y)$ implies $y = 0$). But f is not quasiconcave, as $\{x : f(x) \geqslant 1\}$ is not convex. □

For a proof of the next fact see my notes for Ec 181.

8.3.6 Fact *Let C be a convex set in $\boldsymbol{R}^{\mathrm{m}}$. Let f be a lower semicontinuous quasiconcave function on C that has no local maxima. Then f is explicitly quasiconcave.*

8.3.7 Theorem (Local maxima of explicitly quasiconcave functions) *Let $f\colon C \to \boldsymbol{R}$ be an explicitly quasiconcave function (C convex). If x^* is a local maximizer of f, then it is a global maximizer of f over C.*

Proof: Let x belong to C and suppose $f(x) > f(x^*)$. Then by the definition of explicit quasiconcavity, for any $1 > \lambda > 0$, $f\Big(\lambda x + (1-\lambda)x^*\Big) > f(x^*)$. Since $\lambda x + (1-\lambda)x^* \to x^*$ as $\lambda \to 0$ this contradicts the fact that f has a local maximum at x^*. ∎

8.4 Quasiconcavity and Differentiability

Quasiconcavity has implications for derivatives.

8.4.1 Proposition *Let $C \subset \boldsymbol{R}^{\mathrm{n}}$ be convex and let $f\colon C \to \boldsymbol{R}$ be quasi-concave.*
Notation? Definition?
Let y belong to C and assume that f has a one-sided directional derivative

$$f'(x; y-x) = \lim_{\lambda \downarrow 0} \frac{f\Big(x + \lambda(y-x)\Big) - f(x)}{\lambda}.$$

Then

$$f(y) \geqslant f(x) \implies f'(x; y-x) \geqslant 0.$$

In particular, if f is differentiable at x, then $f'(x) \cdot (y-x) \geqslant 0$ whenever $f(y) \geqslant f(x)$.

Proof: If $f(y) \geqslant f(x)$, then $f\Big(x + \lambda(y-x)\Big) = f\Big((1-\lambda)x + \lambda y\Big) \geqslant f(x)$ for $0 < \lambda \leqslant 1$ by quasiconcavity. Rearranging implies $\frac{f\left(x+\lambda(y-x)\right)-f(x)}{\lambda} \geqslant 0$ and taking limits gives the desired result. ∎

Converse???

8.4.2 Theorem *Let $C \subset \boldsymbol{R}^{\mathrm{n}}$ be open and let $f\colon C \to \boldsymbol{R}$ be quasiconcave and twice-differentiable at $x \in C$. Then*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} D_{i,j}f(x)v_iv_j \leqslant 0 \qquad \text{for any } v \text{ satisfying } f'(x)\cdot v = 0.$$

Proof: Pick $v \in \boldsymbol{R}^{\mathrm{n}}$ and define

$$g(\lambda) = f(x + \lambda v).$$

Then

$$g(0) = f(x), \qquad g'(0) = f'(x)\cdot v, \qquad g''(0) = \sum_{i=1}^{n}\sum_{j=1}^{n} D_{i,j}f(x)v_iv_j.$$

What we have to show is that if $g'(0) = 0$, then $g''(0) \leqslant 0$. Assume for the sake of contradiction that $g'(0) = 0$ and $g''(0) > 0$. Then g has a strict local minimum at zero. That is, for $\varepsilon > 0$ small enough, $f(x+\varepsilon v) > f(x)$ and $f(x-\varepsilon v) > f(x)$. But by quasiconcavity,

$$f(x) = f\Big(\tfrac{1}{2}(x+\varepsilon v) + \tfrac{1}{2}(x-\varepsilon v)\Big) \geqslant \min\{f(x+\varepsilon v), f(x-\varepsilon v)\} > f(x),$$

a contradiction. ∎

References

[1] H. Averch and L. L. Johnson. 1962. Behavior of the firm under regulatory constraint. *American Economic Review* 52(5):1052–1069.
http://www.jstor.org/stable/1812181

[2] A. P. Lerner. 1934. The concept of monopoly and the measurement of monopoly power. *Review of Economic Studies* 1(3):157–175.
http://www.jstor.org/stable/2967480