Caltech Division of the Humanities and Social Sciences

Ec 121a KC Border
Theory of Value Fall 2020

Lecture 4: Production and Returns to Scale

4.1 Production

We now start to worry about where supply comes from.

We start with the special case of a producer that *produces exactly one output from m inputs.*

4.1.1 Production functions

When there is only one output, a **production function** f is often used to describe feasibility. With a production function the inputs as well as the outputs are represented by nonnegative numbers. If $(x_1, \ldots, x_m)$ represent the levels of inputs $1, \ldots, m$, then $f(x_1, \ldots, x_m)$ is the quantity of output generated.

The partial derivative $D_i f(x) = \frac{\partial f(x)}{\partial x_i}$ is the **marginal product** of factor i.

An **isoquant** is just a level curve of the production function f. That is, it is a set of the form

$$\{x \in \boldsymbol{R}^{\mathrm{m}} : f(x) = y\},$$

where y is the level of output. If the production function is **monotonic** and differentiable, then isoquants are surfaces, and we can compute their slope as follows:

For simplicity consider only two inputs, x_1 and x_2. An isoquant implicitly defines x_2 as a function of x_1 via the relation

$$f(x_1, x_2) = y.$$

Let $\hat{x}_2(x_1)$ make this explicit, that is,

$$f\big(x_1, \hat{x}_2(x_1)\big) = y \qquad \text{for all } x_1.$$

The left hand side is now just a function of x_1, and it is a constant function. Therefore its derivative is zero. By the chain rule, then

$$D_1 f + D_2 f \cdot \hat{x}_2' = 0,$$

so

$$\hat{x}_2'(x_1) = -\frac{D_1 f}{D_2 f}\bigg|_{\left(x_1, \hat{x}_2(x_1)\right)}.$$

This is thus the slope of the isoquant. It is also called the **technical rate of substitution**.

4.1.2 Production sets

We now consider a way to describe producers that can potentially produce many commodities.

If there are m commodities, a point y in $\boldsymbol{R}^{\mathrm{m}}$ can be used to represent a **production plan**, where y_i indicates the quantity of good i used or produced. The **sign convention** is that $y_i > 0$ indicates that good i is an output and $y_i < 0$ indicates that it is an input. The **technology set** $Y \subset \boldsymbol{R}^{\mathrm{m}}$ is the set of **feasible** plans.

$$
\begin{array}{rcl}
x \geqq y & \iff & x_i \geqslant y_i,\ i = 1, \ldots, n \\
x > y & \iff & x_i \geqslant y_i,\ i = 1, \ldots, n \text{ and } x \neq y \\
x \gg y & \iff & x_i > y_i,\ i = 1, \ldots, n
\end{array}
$$

Orderings on $\boldsymbol{R}^{\mathrm{n}}$.

A plan $y \in Y$ is **(technologically) efficient** if there is no $y' \in Y$ such that $y' > y$. A **transformation function** is a function $T \colon \boldsymbol{R}^{\mathrm{m}} \to \boldsymbol{R}$ such that $T(y) \geqslant 0$ if and only if $y \in Y$ and $T(y) = 0$ if and only if y is efficient.

4.1.3 Properties

Define

1. **monotonicity**
2. **convexity**
3. **quasiconcavity**
4. Varian's **regularity**, $V(y)$ is closed and nonempty for each $y \geqq 0$.

4.2 Returns to scale

Constant returns to scale means: For all $x \in \boldsymbol{R}^{\mathrm{m}}_{+}$ and all $\lambda > 0$,

1. $f(\lambda x) = \lambda f(x)$.
2. $x \in V(y)$ implies $\lambda x \in V(\lambda y)$.
3. Y is a **cone**. That is, $x \in Y$ implies $\lambda x \in Y$.

Define **increasing returns** and **decreasing returns**.

4.3 Profit Maximization

Now introduce price vectors $p \in \boldsymbol{R}^{\mathrm{m}}_{++}$.

Describe the geometry of the dot product and how it relates to

$$\text{maximize}\, p \cdot y \text{ over } Y.$$

The **optimal profit function** $\pi(p)$. Mention the support function theorem.

4.3.1 Production function approach

Introduce the **wage vector**.

$$\underset{x}{\text{maximize}}\;\; pf(x) - w \cdot x.$$

Let x^* be the optimal input combination, known as the **factor demand function**. The **optimal profit function**

$$\pi(p, w) = pf\Big(x^*(p, w)\Big) - w \cdot x^*(p, w).$$

By the Envelope Theorem we have

$$\frac{\partial \pi}{\partial w_i} = -x_i^*.$$

4.3.2 Leftovers

First order conditions and marginal products.

Exploitation and marginal product. (John Bates Clark)

Define **factor demand functions**.

4.4 Constant returns to scale

Recall that a production function $f \colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$ exhibits **constant returns to scale** if for all $x \in \boldsymbol{R}^{\mathrm{n}}_{+}$ and all $\lambda > 0$,

$$f(\lambda x) = \lambda f(x).$$

Letting $x = 0$ we see that $f(0) = \lambda f(0)$, so $f(0) = 0$.

4.4.1 Homogeneous functions

Let $f\colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$. We say that f is **homogeneous of degree k** if for all $x \in \boldsymbol{R}^{\mathrm{n}}_{+}$ and all $\lambda > 0$,

$$f(\lambda x) = \lambda^k f(x).$$

4.4.1 Euler's theorem *Let $f\colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$ be continuous, and also differentiable on $\boldsymbol{R}^{\mathrm{n}}_{++}$. Then f is homogeneous of degree k if and only if for all $x \in \boldsymbol{R}^{\mathrm{n}}_{++}$,*

$$kf(x) = \sum_{i=1}^{n} D_i f(x) x_i. \qquad (*)$$

4.4.2 Corollary *Let $f\colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$ be continuous and differentiable on $\boldsymbol{R}^{\mathrm{n}}_{++}$. If f is homogeneous of degree k, then $D_j f(x)$ is homogeneous of degree $k-1$.*

4.4.3 Proposition (Everything is constant returns to scale) *Given $f\colon \boldsymbol{R}^{\mathrm{m}} \to \boldsymbol{R}$ define $g\colon \boldsymbol{R}^{\mathrm{m}+1} \to \boldsymbol{R}$ by*

$$g(x_1, \ldots, x_m, z) = zf\Big(\frac{x_1}{z}, \ldots, \frac{x_m}{z}\Big).$$

Then

$$g\Big(\lambda(x,z)\Big) = \lambda z\Big(\frac{\lambda x_1}{\lambda z}, \ldots, \frac{\lambda x_m}{\lambda z}\Big) = \lambda\left\{zf\Big(\frac{x_1}{z}, \ldots, \frac{x_m}{z}\Big)\right\} = \lambda g(x,z).$$

4.4.2 Quasiconcavity and Constant Returns to Scale

The next result has applications to production functions. (Cf. Jehle [4, Theorem 5.2.1, pp. 224–225] and Shephard [5, pp. 5–7].)

4.4.4 Theorem *Let $f\colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}_{+}$ be nonnegative, nondecreasing, quasiconcave, and positively homogeneous of degree k where $0 < k \leqslant 1$. Then f is concave.*

Proof: Let $x, y \in \boldsymbol{R}^{\mathrm{n}}$ and suppose first that $f(x) = \alpha > 0$ and $f(y) = \beta > 0$. (The case $\alpha = 0$ and/or $\beta = 0$ will be considered in a moment.) Then by homogeneity,

$$f\Big(\frac{x}{\alpha^{\frac{1}{k}}}\Big) = f\Big(\frac{y}{\beta^{\frac{1}{k}}}\Big) = 1$$

By quasiconcavity,

$$f\Big(\lambda\frac{x}{\alpha^{\frac{1}{k}}} + (1-\lambda)\frac{y}{\beta^{\frac{1}{k}}}\Big) \geqslant 1$$

for $0 \leqslant \lambda \leqslant 1$. So setting $\lambda = \frac{\alpha^{\frac{1}{k}}}{\alpha^{\frac{1}{k}}+\beta^{\frac{1}{k}}}$, we have

$$f\Big(\frac{x}{\alpha^{\frac{1}{k}}+\beta^{\frac{1}{k}}} + \frac{y}{\alpha^{\frac{1}{k}}+\beta^{\frac{1}{k}}}\Big) \geqslant 1.$$

By homogeneity,

$$f(x+y) \geqslant (\alpha^{\frac{1}{k}} + \beta^{\frac{1}{k}})^k = \Big[f(x)^{\frac{1}{k}} + f(y)^{\frac{1}{k}}\Big]^k. \tag{1}$$

Observe that since f is nonnegative and nondecreasing, (1) holds even if $f(x) = 0$ or $f(y) = 0$. Now replace x by μx and y by $(1-\mu)y$ in (1), where $0 \leqslant \mu \leqslant 1$, to get

$$\begin{aligned} f\Big(\mu x + (1-\mu)y\Big) &\geqslant \Big[f(\mu x)^{\frac{1}{k}} + f\Big((1-\mu)y\Big)^{\frac{1}{k}}\Big]^k \\ &= \Big[\mu f(x)^{\frac{1}{k}} + (1-\mu)f(y)^{\frac{1}{k}}\Big]^k \\ &\geqslant \mu\Big(f(x)^{\frac{1}{k}}\Big)^k + (1-\mu)\Big(f(y)^{\frac{1}{k}}\Big)^k \\ &= \mu f(x) + (1-\mu)f(y), \end{aligned}$$

where the last inequality follows from the concavity of $\gamma \mapsto \gamma^k$. Since x and y are arbitrary, f is concave. ∎

4.4.3 An application to the Cobb–Douglas function

4.4.5 Proposition *The Cobb–Douglas function defined by*

$$f(x_1,\dots,x_n) = x_1^{\alpha_1}\, x_2^{\alpha_2}\, \cdots\, x_n^{\alpha_n},$$

where $\alpha_i > 0$, $i = 1,\dots,n$, *and* $\sum_i \alpha_i \leqslant 1$, *is a concave function.*

Proof: Start by observing the extended-real valued function $x \mapsto \ln x$ is strictly concave on $\boldsymbol{R}_+$, since its second derivative is everywhere strictly negative. Therefore the function $(x_1,\dots,x_n) \mapsto \ln x_i$ is concave on $\boldsymbol{R}^{\mathrm{n}}_+$ for each i. Since nonnegative scalar multiples and sums of concave functions are concave, the function

$$\varphi\colon (x_1,\dots,x_n) \mapsto \sum_{i=1}^n \alpha_i \ln x_i$$

is concave and therefore quasiconcave. Now the function $y \mapsto e^y$ is strictly monotonic, so its composition with φ, namely

$$f(x_1,\dots,x_n) = e^{\varphi(x)} = x_1^{\alpha_1}\, x_2^{\alpha_2}\, \cdots\, x_n^{\alpha_n},$$

is quasiconcave. But f is homogeneous of degree $\alpha = \alpha_1 + \cdots + \alpha_n \leqslant 1$, so by Theorem 4.4.4, it is concave. ∎

4.5 Constant returns and Profit Maximization

An important and somewhat counterintuitive property of constant returns to scale production is this.

> If a production function f exhibits constant returns to scale and if the problem
>
> $$\underset{x}{\text{maximize}}\, \pi(x) = pf(x) - w \cdot x$$
>
> has a solution, the the optimal profit is zero.

The proof is simple. By constant returns $f(0) = f(0x) = 0f(x) = 0$ for any x, so $f(0) = 0$, and it thus always possible to earn a profit of zero by setting $x = 0$. On the other hand if $pf(\bar{x}) - w \cdot \bar{x} > 0$, then no profit maximizer can exist, because if $\pi(\bar{x}) > 0$, then $\pi(2\bar{x}) = 2\pi(\bar{x}) > \pi(\bar{x})$.

So the only way a profit maximizer can exist is if the maximal profit is zero. This implies a very special relationship between p and $w_1, \dots, w_n$ must exist. For instance, in the case of one input ($n = 1$), constant returns to scale and monotonicity imply that $f(x)$ is of the form $f(x) = \alpha x$ for some $\alpha > 0$. Then the producer wants to maximize

$$p\alpha x - wx = (\alpha p - w)x$$

over the interval $[0, \infty)$. This is a linear function of x and achieves a unique maximum at $x = 0$ if $\alpha p < w$, and if $\alpha p = w$, then every $x \geqslant 0$ maximizes profit. In this case, the supply curve is vertical at $p = w/\alpha$, so it isn't really a supply *function*. Instead we call it a supply correspondence, and it is undefined for $p > \alpha/w$.

More generally, if there are $n > 1$ inputs and $x^* \gg 0$ maximizes profit, the first order conditions tell us that

$$p\frac{\partial f(x^*)}{\partial x_i} = w_i, \quad i = 1, \dots, n,$$

so and Euler's theorem yields

$$pf(x^*) = p\sum_{i=1}^{n} \frac{\partial f(x^*)}{\partial x_i} x_i^* = \sum_{i=1}^{n} w_i x_i^* = w \cdot x^*,$$

again the profit $pf(x^*) - w \cdot x^* = 0$.

4.6 Per capita analysis and macroeconomics

In a simple model of the macroeconomy, there is one good, **output**, which may either be consumed or saved to become part of the **capital stock** K. Output is produced from capital and labor according to the aggregate production function F,

$$Y = F(K, L),$$

Y is the flow of real output, and K is the capital stock, and L is the flow of labor supply. If F exhibits constant return to scale,

$$F(\lambda K, \lambda L) = \lambda F(K, L),$$

Euler's theorem tells us that

$$F_K(K, L)K + F_L(K, L)L = F(K, L).$$

We may also analyze the economy in per capita terms. Define

$$y = \frac{Y}{L} \qquad k = \frac{K}{L}.$$

Then

$$y = \frac{F(K, L)}{L} = F(K/L, \underbrace{L/L}_{=1}) = f(k).$$

Savings and Population Dynamics

We now make everything a function of time t. Start by assuming a constant rate of growth of the labor supply:

$$\frac{\dot{L}(t)}{L(t)} = n \qquad \text{or} \qquad L(t) = L_0 e^{nt}$$

where the dot denotes differentiation with respect to time t, and n is an exogenous constant. If there is no depreciation of capital, and a constant fraction s of output is saved, then

$$\dot{K}(t) = sY(t).$$

We may write K in terms of k as

$$K(t) = k(t)L(t)$$

which implies

$$\dot{K} = \dot{k}L + nkL.$$

But we may also write $\dot{K}$ in terms of Y as

$$\dot{K}(t) = sY(t) = sL(t)f\big(k(t)\big)$$

Combining these two expressions for $\dot{K}$ gives

$$(\dot{k} + nk)L = sLf(k)$$

or

$$\dot{k} + nk = sf(k)$$

So

$$\dot{k}(t) = sf\Big((k(t)\Big) - nk(t) \tag{2}$$

and

$$\dot{y}(t) = f'\Big((k(t)\Big)\dot{k}(t).$$

Example: Cobb–Douglas Production (Solow)

For the case

$$F(K, L) = K^\alpha L^{1-\alpha}$$

we have

$$f(k) = k^\alpha$$

so (2) becomes

$$\dot{k} = sk^\alpha - nk.$$

The solution to this differential equation is

$$k(t) = \left[\left(k_0^{1-\alpha} - \frac{s}{n}\right) e^{-n(1-\alpha)t} + \frac{s}{n}\right]^{\frac{1}{1-\alpha}}$$

So

$$k(t) \to k^* = \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}}$$
$$y(t) \to y^* = \left(\frac{s}{n}\right)^{\frac{\alpha}{1-\alpha}}$$

And growth stops.

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