Caltech Division of the Humanities and Social Sciences

Ec 121a KC Border
Theory of Value Fall 2020

Homework I: Profit Maximization and such

When you write up the answers, please use complete sentences and explain what you are doing—what is obvious to you may not be obvious to me or anyone else. A good idea is to try to write as if you are writing a textbook.

Also, please display your final results by boxing them, just as these paragraphs are boxed.

Due 4:00 pm Thursday, October 15 in Canvas.

1 (70 pts)

A producer produces y units of output using a single input x according to the production function

$$y = x^{\alpha}, \qquad x \geqslant 0$$

where $\alpha > 0$. The producer may sell as much as he wants at a unit price p, but must pay a wage w for every unit of x used. The producer chooses x to maximize profits.

1. (5pts) Write down the producer's objective function.
2. (5 pts) What are the necessary first and second order conditions for the profit maximization problem?
3. (10 pts) For which values of α can the first and second order conditions be satisfied? (In other words, for which values of α must these conditions fail?)
4. (15 pts) Use the first order necessary conditions to derive the expression for the profit maximizing level of output $y^*(p, w)$, the profit maximizing level of input $x^*(p, w)$, and the optimal profit $\pi^*(p, w) = py^*(p, w) - wx^*(p, w)$.
5. (15 pts) What are the comparative statics of this problem? That is, what are the partial derivatives of the functions y^* and x^* with respect to p and w? What are their signs?
6. (10 pts) When $\alpha = 1$, what relation between p and w must be satisfied so that the profit maximizing level of output is positive?
7. (10 pts) Compute the partial derivatives of π^* with respect to p and w. Relate them to y^* and x^*. Does this agree with the Envelope Theorem? (Hint: If it doesn't you've done something wrong.)

2 (30 pts)

A monopolist produces output and sells it in two distinct markets, with revenue functions R_1 and R_2. The total cost of producing the output y_1 for market 1 and y_2 for market 2 is $C(y_1+y_2)$. (Here C is a function, not a constant.)

1. (5 pts) Show that a profit maximizing monopolist equates marginal revenue in each market to the common marginal cost.

2. (5 pts) What can we say about the slopes of the marginal revenue and marginal cost functions in each market? (Hint: Use the second order conditions.)

3. (5 pts) Given a revenue function $R(y)$, the average price for selling y units of output is $p(y) = \frac{R(y)}{y}$. This function is called the *inverse demand function*, and gives the price as a function of the quantity sold. The *elasticity of demand* η is defined by

$$\eta(y) = \frac{-p(y)}{yp'(y)}.$$

   Note that if the demand curve p is downward sloping, then the elasticity is a positive number.

   Show that $R'(y) = p(1 - \frac{1}{\eta})$.

4. (5 pts) Show that the price is higher in the market with the less elastic demand.

5. (10 pts) Suppose the output of the monopolist in market 1 is taxed at a rate t per unit sold, paid by the monopolist. From the hypothesis of profit maximization alone, can we sign the change in the output in either market? What about the change in the total output? What plausible additional assumptions can you make that will allow you to sign all three changes? What are the signs?

3 Convex and concave functions

Recall that a function $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ is **convex** if its domain is convex and it satisfies

$$f\big(tx + (1-t)y\big) \leqslant tf(x) + (1-t)f(y)$$

for all x, y in the domain and all $t \in (0,1)$. It is **concave** if its domain is convex and it satisfies

$$f\big(tx + (1-t)y\big) \geqslant tf(x) + (1-t)f(y)$$

for all x, y in the domain and all $t \in (0,1)$.

It is a fact that a twice differentiable function is convex if and only if its Hessian matrix

$$\big[D_{ij}f(x)\big] = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right]$$

is positive semidefinite for all x. It is concave if and only if the Hessian matrix is negative semidefinite everywhere.

There are two useful tests for semidefiniteness. A matrix is positive semidefinite if and only all its eigenvalues are nonnegative. A matrix is negative semidefinite if and only all its eigenvalues are nonpositive. A matrix is positive definite if and only all its eigenvalues are strictly positive, and negative definite if and only if the eigenvalues are all strictly negative.

The other useful test is based on principal minors. The **NW principal minors** of the $n \times n$ matrix A are the n determinants

$$\begin{vmatrix} a_{11} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots, \begin{vmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \ldots & a_{kk} \end{vmatrix}, \ldots, \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix}.$$

The matrix A is positive definite if and only if all its NW principal minors are strictly positive. The matrix A is negative definite if and only if $-A$ is positive definite, so given the rules for determinants, if and only if

$$\begin{vmatrix} a_{11} \end{vmatrix} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots, (-1)^k \begin{vmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \ldots & a_{kk} \end{vmatrix} > 0, \ldots, (-1)^n \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} > 0.$$

3.1 Definiteness

(28 pts) Categorize the following matrices by (semi)definiteness or indefiniteness. Explain your reasons.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 7 & 0 \\ 7 & 4 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

3.2 Convex and concave functions I

(42 pts) Decide whether the following functions are concave, convex, or neither. Explain your reasons. Sketch the graph of each.

1. $f(x) = \ln(1 + x), \quad x \geqslant 0.$

2. $f(x) = \ln x, \quad x > 0.$

3. $f(x) = e^x, \quad x \in \boldsymbol{R}.$

4. $f(x) = \dfrac{x}{1 + x}, \quad x \geqslant 0.$

5. $f(x) = \dfrac{e^x}{1 + e^x}, \quad x \in \boldsymbol{R}.$

6. $f(x) = x^\alpha, \quad x \geqslant 0.$ (Hint: The answer depends on α.)

7. $f(x) = x^\alpha - x, \quad x \geqslant 0.$

3.3 Convex and concave functions II

(40 pts) For which positive values of α and β is the function

$$f(x, y) = x^{\alpha} y^{\beta}$$

concave on $\boldsymbol{R}^2_+$? Why?