

1 The space \mathcal{N} of natural sequences

Another fundamental metric space is the space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ of sequences of natural numbers, endowed with its product topology. Since the discrete metric on \mathbb{N} is complete, \mathbb{N} is a Polish space. Thus, Theorem ?? guarantees that \mathcal{N} is Polish too. As a countably infinite product of a countable space, countable products of \mathcal{N} all look alike, as the next theorem asserts.

Theorem 1 (Products of \mathcal{N}) *The Polish spaces $\mathcal{N}^2, \mathcal{N}^3, \dots$, and $\mathcal{N}^{\mathbb{N}}$ are all homeomorphic to \mathcal{N} .*

Proof: We prove only that \mathcal{N} and $\mathcal{N}^{\mathbb{N}}$ are homeomorphic. You can easily see how to modify this proof to show that \mathcal{N} and \mathcal{N}^n are homeomorphic.

Choose an infinite countable partition $\{N_1, N_2, \dots\}$ of \mathbb{N} such that each N_k is countably infinite. That is, $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$, $N_i \cap N_j = \emptyset$ for $i \neq j$, and each N_i is infinite. Let $N_i = \{k_1^i, k_2^i, \dots\}$ for each i . Now consider the function $f: \mathcal{N} \rightarrow \mathcal{N}^{\mathbb{N}}$ defined by $f(\mathbf{n}) = (\mathbf{n}_1, \mathbf{n}_2, \dots)$, where $\mathbf{n}_i = (n_{k_1^i}, n_{k_2^i}, \dots)$. Clearly, f is surjective and one-to-one. We leave the task of verifying that f is a homeomorphism as an exercise. ■

As an aside, we mention the surprising fact that \mathcal{N} is also homeomorphic to the space of irrational numbers in $(0, 1)$ (with their usual metric topology). Hence \mathcal{N} has the cardinality of the continuum. In order to prove this, we must develop the theory of infinite continued fractions, which is not widely taught these days. Our exposition is based on C. D. Olds [?].

Theorem 2 (The space of irrationals) *The Polish space \mathcal{N} is homeomorphic to the metric space \mathcal{J} of irrationals in $(0, 1)$, equipped with the usual absolute value metric.*

Proof: We start by describing an algorithm for mapping \mathcal{J} into \mathcal{N} . Given $x \in \mathcal{J}$, set $x_0 = x$. Note that $\frac{1}{x_0} > 1$. Recursively define x_m and n_m by $\frac{1}{x_m} = n_{m+1} + x_{m+1}$, where $n_{m+1} = \lfloor \frac{1}{x_m} \rfloor$, the largest integer less than or equal to $\frac{1}{x_m}$, and $x_{m+1} \in (0, 1)$ is irrational. Notice that this process does not terminate. (Why?) Then $x = x_0 = \frac{1}{n_1 + x_1}$, $x_1 = \frac{1}{n_2 + x_2}$, etc. Consequently, $x = \frac{1}{n_1 + x_1} = \frac{1}{n_1 + \frac{1}{n_2 + x_2}}$, etc. We can express this sequence of equalities formally as an infinite **continued fraction**:

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{\ddots}}}}. \quad (1)$$

This algorithm assigns to each irrational x in $(0, 1)$ a unique sequence $f(x) = (n_1, n_2, \dots)$ belonging to \mathcal{N} . It is easy to see that if a sequence $\{x_m\}$ of irrationals converges to an irrational x , then $f(x_m)$ converges to $f(x)$ pointwise in \mathcal{N} .

To see that f is surjective, we now describe an algorithm that inverts the above process. This algorithm gives a meaning to the infinite continued fraction (1) above. We start by recursively defining several sequences of functions. Let P denote the set of strictly positive real numbers. For each k , define $c_k: P^k \rightarrow P$ recursively by $c_1(a_1) = \frac{1}{a_1}$, and for $k \geq 1$, define

$$c_{k+1}(a_1, \dots, a_k, a_{k+1}) = c_k\left(a_1, \dots, a_{k-1}, a_k + \frac{1}{a_{k+1}}\right). \quad (2)$$

The value of c_k is a “ k -decked” continued fraction. Define $p_k, q_k: P^k \rightarrow P$ by $p_1(a_1) = 1$, $q_1(a_1) = a_1$, $p_2(a_1, a_2) = a_1 a_2$, $q_2(a_1, a_2) = a_1 a_2 + 1$, and for $k \geq 2$, recursively define

$$p_{k+1}(a_1, \dots, a_{k+1}) = a_{k+1} p_k(a_1, \dots, a_k) + p_{k-1}(a_1, \dots, a_{k-1}) \quad (3)$$

and

$$q_{k+1}(a_1, \dots, a_{k+1}) = a_{k+1} q_k(a_1, \dots, a_k) + q_{k-1}(a_1, \dots, a_{k-1}). \quad (4)$$

(The functions p_k and q_k obey the same difference equation, but have different initial conditions.) Observe that we have chosen p_1 , p_2 , q_1 , and q_2 so that $c_1(a_1) = \frac{p_1(a_1)}{q_1(a_1)}$ and $c_2(a_1, a_2) = \frac{p_2(a_1, a_2)}{q_2(a_1, a_2)}$ for any (a_1, a_2) . We now proceed by induction to show that

$$c_k(a_1, \dots, a_k) = \frac{p_k(a_1, \dots, a_k)}{q_k(a_1, \dots, a_k)} \quad (5)$$

for every k , and every sequence $\{a_n\}$ of strictly positive reals. So suppose that (5) holds for k . Then:

$$\begin{aligned} c_{k+1}(a_1, \dots, a_{k+1}) &= c_k\left(a_1, \dots, a_{k-1}, a_k + \frac{1}{a_{k+1}}\right) \\ &= \frac{p_k\left(a_1, \dots, a_{k-1}, a_k + \frac{1}{a_{k+1}}\right)}{q_k\left(a_1, \dots, a_{k-1}, a_k + \frac{1}{a_{k+1}}\right)} \\ &= \frac{\left(a_k + \frac{1}{a_{k+1}}\right)p_{k-1}(a_1, \dots, a_{k-1}) + p_{k-2}(a_1, \dots, a_{k-2})}{\left(a_k + \frac{1}{a_{k+1}}\right)q_{k-1}(a_1, \dots, a_{k-1}) + q_{k-2}(a_1, \dots, a_{k-2})} \\ &= \frac{(a_k a_{k+1} + 1)p_{k-1}(a_1, \dots, a_{k-1}) + a_{k+1}p_{k-2}(a_1, \dots, a_{k-2})}{(a_k a_{k+1} + 1)q_{k-1}(a_1, \dots, a_{k-1}) + a_{k+1}q_{k-2}(a_1, \dots, a_{k-2})} \\ &= \frac{a_{k+1} [a_k p_{k-1}(a_1, \dots, a_{k-1}) + p_{k-2}(a_1, \dots, a_{k-2})] + p_{k-1}(a_1, \dots, a_{k-1})}{a_{k+1} [a_k q_{k-1}(a_1, \dots, a_{k-1}) + q_{k-2}(a_1, \dots, a_{k-2})] + q_{k-1}(a_1, \dots, a_{k-1})} \\ &= \frac{a_{k+1} p_k(a_1, \dots, a_k) + p_{k-1}(a_1, \dots, a_{k-1})}{a_{k+1} q_k(a_1, \dots, a_k) + q_{k-1}(a_1, \dots, a_{k-1})} \\ &= \frac{p_{k+1}(a_1, \dots, a_{k+1})}{q_{k+1}(a_1, \dots, a_{k+1})}, \end{aligned}$$

where the last two equalities follows from the recursion relations (3) and (4). This completes the induction argument.

Next we derive the recursion relation

$$p_k(a_1, \dots, a_k)q_{k-1}(a_1, \dots, a_{k-1}) - p_{k-1}(a_1, \dots, a_{k-1})q_k(a_1, \dots, a_k) = (-1)^{k+1}, \quad (6)$$

for $k = 0, 1, \dots$, where we artificially define $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$. (This convention is consistent with our recursive definition above for $k = 1, 2$.) The proof is again by induction. The case $k = 0$ is verified by direct calculation. So assume that the formula (6) is true for k . Then

$$\begin{aligned} & p_{k+1}(a_1, \dots, a_{k+1})q_k(a_1, \dots, a_k) - p_k(a_1, \dots, a_k)q_{k+1}(a_1, \dots, a_{k+1}) \\ &= [a_{k+1}p_k(a_1, \dots, a_k) + p_{k-1}(a_1, \dots, a_{k-1})]q_k(a_1, \dots, a_k) \\ &\quad - p_k(a_1, \dots, a_k)[a_{k+1}q_k(a_1, \dots, a_k) + q_{k-1}(a_1, \dots, a_{k-1})] \\ &= (-1)[p_kq_{k-1} - p_{k-1}q_k] = (-1)^{(k+1)+1}, \end{aligned}$$

where the last equality follows from the induction hypothesis. This completes this induction argument.

Given a point $(n_1, n_2, \dots) \in \mathcal{N}$, the numbers $c_1(n_1)$, $c_2(n_1, n_2)$, etc., are called the convergents of the continued fraction (1). (Note that since each $n_j \geq 1$, for $k > 1$ we have $p_k(n_1, \dots, n_k) > p_{k-1}(n_1, \dots, n_{k-1})$ and $q_k(n_1, \dots, n_k) > q_{k-1}(n_1, \dots, n_{k-1})$.) Using (5) and (6), we see that

$$c_k - c_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k+1}}{q_k q_{k-1}}. \quad (7)$$

We also have that

$$\begin{aligned} c_k - c_{k-2} &= \frac{p_k}{q_k} - \frac{p_{k-2}}{q_{k-2}} \\ &= \frac{p_k q_{k-1} - p_{k-2} q_k}{q_k q_{k-1}} \\ &= \frac{q_k q_{k-2}}{(n_k p_{k-1} + p_{k-2})q_{k-1} - p_{k-2}(n_k q_{k-1} + q_{k-2})} \\ &= \frac{q_k q_{k-2}}{n_k(p_{k-1} q_{k-2} - p_{k-2} q_{k-1})} \\ &= n_k \frac{(-1)^k}{q_k q_{k-2}}. \end{aligned} \quad (8)$$

Equations (7) and (8) in turn imply that

$$0 < c_2 < c_4 < c_6 < \dots < c_5 < c_3 < c_1 < 1, \quad (9)$$

and that

$$c_k(n_1, \dots, n_k) - c_{k-1}(n_1, \dots, n_{k-1}) \xrightarrow[k \rightarrow \infty]{} 0.$$

Therefore there is some $x \in \mathbf{R}$ with $c_k(n_1, \dots, n_k) \xrightarrow[k \rightarrow \infty]{} x$. We claim that for this x the first algorithm yields $f(x) = (n_1, n_2, \dots)$. To see this, observe that if x is the specified limit, then (9) implies that $\frac{1}{n_1 + \frac{1}{n_2}} < x < \frac{1}{n_1}$. From this we see

that if we write $x = \frac{1}{n_1+x_1}$, then it must be the case that $0 < x_1 < \frac{1}{n_2} \leq 1$, so that n_1 is the first term generated by our first algorithm. Proceeding inductively, you can see that $f(x) = (n_1, n_2, \dots)$, as desired.

There is one point left. How do we know that the x generated by the second algorithm is irrational? The answer is that if that if x is rational, eventually the first algorithm stops, so it cannot generate an infinite sequence of natural numbers. (Remember, zero is not a natural number here.) To see this, recall the familiar Euclidean algorithm

$$\frac{q}{p} = n_1 + \frac{r_1}{p}, \quad \frac{p}{r_1} = n_2 + \frac{r_2}{r_1}, \quad \frac{r_1}{r_2} = n_3 + \frac{r_3}{r_2}, \dots,$$

where $r_i \geq 0$ and $r_i < r_{i-1}$. Hence, if x is rational, r_i must eventually be zero, so the process stops after a finite number of steps. Thus x is irrational.

Finally, we leave the verification that f is indeed a homeomorphism as an exercise. ■

Since we can write $J = \bigcap_{q \text{ rational}} \{q\}^c$, we see that J is a \mathcal{G}_δ in \mathbf{R} . Therefore Alexandroff's Lemma ?? implies that it is completely metrizable, even though the usual metric is not complete. The result above describes another complete metric on J .

Just as Theorem ?? showed that every compact metrizable set is a continuous image of the Cantor set, every Polish space is a continuous image of \mathcal{N} . A naive attempt to show this might run like this. In a separable space X , we can fix a countable dense subset $D = \{x_1, x_2, \dots\}$. If X is metrizable, any point in X is the limit of a sequence in D . We can in the natural way identify this sequence with an element of \mathcal{N} . (Identify the sequence $\{x_{n_1}, x_{n_2}, \dots\}$ of points in D with the point (n_1, n_2, \dots) in \mathcal{N} .) Thus, we can view X as the image of a subset of \mathcal{N} . But clearly our naive approach will not work, since not every sequence in D is convergent. The proof of the next theorem presents a precise construction along these lines of thought that actually works.

Theorem 3 *Every Polish space is a continuous image of \mathcal{N} .*

Proof: Let X be a Polish space, and let d be a compatible complete metric. Since X is separable, there is a countably infinite family $\{F_1, F_2, \dots\}$ of (not necessarily pairwise distinct) nonempty closed subsets of X , each with $\text{diam } F_n \leq 1$, and satisfying $X = \bigcup_{n=1}^{\infty} F_n$. Then each F_n is also a Polish space, so it may be written as a countably infinite union in a similar fashion, but this time requiring that the diameter of the closed sets be no more than $\frac{1}{2}$. We repeat this process ad infinitum, so that for every finite sequence (n_1, n_2, \dots, n_m) of natural numbers we have selected a closed subset F_{n_1, n_2, \dots, n_m} of X with $\text{diam } F_{n_1, n_2, \dots, n_m} \leq \frac{1}{m}$, and also satisfying

$$F_{n_1, n_2, \dots, n_m} = \bigcup_{k=1}^{\infty} F_{n_1, n_2, \dots, n_m, k}. \quad (\star)$$

(A family of sets satisfying (\star) indexed by finite sequences of natural numbers is called a **Suslin scheme**. If it satisfies the additional properties of our family,

closedness and diameters decreasing to zero, it is sometimes called a **Polish** scheme.)

Now let $\mathbf{n} = (n_1, n_2, \dots) \in \mathcal{N}$ be a sequence of natural numbers. Then notice that the sequence of closed sets $\{F_{n_1, \dots, n_k} : k = 1, 2, \dots\}$ satisfies

$$F_{n_1} \supset F_{n_1, n_2} \supset \dots \supset F_{n_1, \dots, n_k} \supset \dots$$

and $\lim_{k \rightarrow \infty} \text{diam } F_{n_1, \dots, n_k} = 0$. Since X is d -complete, it follows from Theorem ?? that $\bigcap_{k=1}^{\infty} F_{n_1, \dots, n_k}$ is a singleton. Thus, we can define a mapping $f: \mathcal{N} \rightarrow X$, via $\mathbf{n} \mapsto f(\mathbf{n})$, where $\{f(\mathbf{n})\} = \bigcap_{k=1}^{\infty} F_{n_1, \dots, n_k}$. Since each point of X belongs to some F_{n_1} , and so to some F_{n_1, n_2} , etc., it is easy to see that the mapping $f: \mathcal{N} \rightarrow X$ is also surjective.

The continuity of f is easy to verify. Fix $\mathbf{k} = (k_1, k_2, \dots) \in \mathcal{N}$ and let $\varepsilon > 0$ be given. Choose some m such that $\frac{1}{m} < \frac{\varepsilon}{2}$. Then the set

$$U = \{\mathbf{n} \in \mathcal{N} : n_1 = k_1, \dots, n_m = k_m\}$$

is a neighborhood of \mathbf{k} in \mathcal{N} (why?), and satisfies $f(U) \subset F_{k_1, k_2, \dots, k_m}$. Since $\text{diam } F_{k_1, k_2, \dots, k_m} \leq \frac{1}{m} < \frac{\varepsilon}{2}$, if $\mathbf{n} \in U$, then $d(f(\mathbf{n}), f(\mathbf{k})) < \varepsilon$. That is, f is continuous at \mathbf{k} , and the proof is finished. ■