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Continuity of the coordinate map

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An “obvious” fact

1 Theorem *Let* $\{x_1,\dots,x_m\}$ *be a linearly independent subset of* $\boldsymbol{R}^{\mathrm{p}}$*. Let* α_n *be a sequence in* $\boldsymbol{R}^{\mathrm{m}}$*. If* $\sum_{j=1}^m \alpha_{nj}x_j \xrightarrow[n\to\infty]{} \sum_{j=1}^m \alpha_j x_j$*, then for each* $j = 1,\dots,m$*, we have* $\alpha_{nj} \xrightarrow[n\to\infty]{} \alpha_j$*.*

Proof: Let X be the $p \times m$ matrix whose j^{th} column is x_j. Then a linear combination $x = \sum_{j=1}^m \alpha_j x_j$ of $\{x_1,\dots,x_m\}$ can be written as $X\alpha$, where α is the column vector of coordinates $\alpha_1,\dots,\alpha_m$. By the theory of ordinary least squares estimation, the coordinate mapping $T(x)\colon x \mapsto \alpha$ is given by

$$T(x) = (X'X)^{-1}X'x,$$

which is clearly continuous. ∎

Alternate proof: (Franklin [2, pp. 54–55]) Since $\{x_1,\dots,x_m\}$ is a linearly independent set, $(\beta_1,\dots,\beta_m) \neq 0$ implies that $\sum_{j=1}^m \beta_j x_j \neq 0$. Consider the unit sphere in $\boldsymbol{R}^{\mathrm{m}}$,

$$S = \Big\{(\beta_1,\dots,\beta_m) \in \boldsymbol{R}^{\mathrm{m}} : \sum_{j=1}^m \beta_j^2 = 1\Big\}.$$

This is a compact subset of $\boldsymbol{R}^{\mathrm{m}}$ so the continuous function

$$(\beta_1,\dots,\beta_m) \mapsto \Big\|\sum_{j=1}^m \beta_j x_j\Big\|$$

attains a minimum μ on S. Moreover this minimum μ is not zero. (For if $\|\sum_{j=1}^m \beta_j x_j\| = 0$, then $\sum_{j=1}^m \beta_j x_j = 0$, which implies $\beta_j = 0$, $j = 1,\dots,m$ by linear independence, which contradicts $\sum_{j=1}^m \beta_j^2 = 1$.)[1] Thus by homogeneity of the norm,

$$\sum_{j=1}^m \beta_j^2 \geqslant \rho^2 \implies \Big\|\sum_{j=1}^m \beta_j x_j\Big\| \geqslant \rho\mu. \tag{1}$$

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[1] N.B. This is where we use the assumption of linear independence.

So assume that $\sum_{j=1}^m \alpha_{nj} x_j \xrightarrow[n\to\infty]{} \sum_{j=1}^m \alpha_j x_j$. Then $\sum_{j=1}^m (\alpha_{nj} - \alpha_j) x_j \xrightarrow[n\to\infty]{} 0$. Thus for every $\varepsilon > 0$, there is some n_ε such that for all $n \geqslant n_\varepsilon$, we have

$$\Big\| \sum_{j=1}^{m} (\alpha_{nj} - \alpha_j) x_j \Big\| < \varepsilon,$$

so by the contrapositive of (1) we have

$$\sum_{j=1}^{m} (\alpha_{nj} - \alpha_j)^2 < \left(\frac{\varepsilon}{\mu}\right)^2.$$

This implies that $\sum_{j=1}^m (\alpha_{nj} - \alpha_j)^2 \xrightarrow[n\to\infty]{} 0$, so for each $j = 1, \ldots, m$ we have $\alpha_{nj} \to \alpha_j$. ∎

Note that the above proof also proves Theorem 1 for any normed vector space in place of $\boldsymbol{R}^{\mathrm{p}}$.

Despite the fact that many authors apparently believe that the result is trivial, it is in fact rather delicate—it fails if $\{x_1, \ldots, x_m\}$ is linearly dependent.

2 Example (Theorem 1 fails without linear independence) Let $x_1 = (1, 1)$ and $x_2 = (-1, -1)$ in $\boldsymbol{R}^2$. Then $nx_1 + nx_2 = (0, 0) \to (0, 0) = 0x_1 + 0x_2$, but $n \not\to 0$.

An even more trivial example is to let $x_1 = (0, 0)$, remembering that $\{(0, 0)\}$ is linearly dependent. Then $n(0, 0) \to 42(0, 0)$, but $n \not\to 42$. □

Generalization

3 Theorem *Let X be a Hausdorff topological vector space of dimension m, and let $x_1, \ldots, x_m$ be an ordered basis for X. The* ***coordinate mapping*** *$T\colon X \to \boldsymbol{R}^{\mathrm{m}}$ defined by*

$$T\Big(\sum_{i=1}^{m} \alpha_i x_i\Big) = (\alpha_1, \ldots, \alpha_m)$$

is a linear homeomorphism. That is, T is well-defined, linear, one-to-one, maps X onto $\boldsymbol{R}^{\mathrm{m}}$, is continuous, and T^{-1} is continuous.

Proof: We know from basic linear algebra that T is a linear bijection from X onto $\boldsymbol{R}^{\mathrm{m}}$. We also know that the inverse mapping $T^{-1}\colon (\alpha_1, \ldots, \alpha_m) \mapsto \sum_{i=1}^m \alpha_i x_i$ is a linear bijection. Moreover, T^{-1} is continuous, as scalar multiplication and vector addition are continuous. It remains to prove that T is continuous. It suffices to prove that T is continuous at zero.

Let B be the open unit ball and let S be the unit sphere in $\boldsymbol{R}^{\mathrm{m}}$. Since S is compact and T^{-1} is continuous, $T^{-1}(S)$ is compact. Since X is Hausdorff, $T^{-1}(S)$ is closed. Now $0_X \notin T^{-1}(S)$, as $0_{\boldsymbol{R}^{\mathrm{m}}} \notin S$, so there exists a circled neighborhood V of zero such that $V \cap T^{-1}(S) = \varnothing$. Since V is circled, we have $V \subset T^{-1}(B)$: For if there exists some $x \in V$ such that $x \notin T^{-1}(B)$ (that is, $\|T(x)\| \geq 1$), then $\frac{x}{\|T(x)\|} \in V \cap T^{-1}(S)$, a contradiction.

Thus, $T^{-1}(B)$ is a neighborhood of zero. Since scalar multiples of B form a neighborhood base at zero in $\boldsymbol{R}^{\mathrm{m}}$, we see that T is continuous at zero, and therefore continuous. ∎

Informally this says that $\boldsymbol{R}^{\mathrm{m}}$ is the *only* m-dimensional Hausdorff topological vector space. A useful corollary of this is the following.

4 Corollary *Let X be a Hausdorff topological vector space, and let $\{x_1, \ldots, x_m\}$ be a linearly independent subset of X. Let α_n be a sequence in $\boldsymbol{R}^{\mathrm{m}}$. If $\sum_{j=1}^m \alpha_{nj} x_j \xrightarrow[n\to\infty]{} \sum_{j=1}^m \alpha_j x_j$, then for each $j = 1, \ldots, m$, we have $\alpha_{nj} \xrightarrow[m\to\infty]{} \alpha_j$.*

Proof: The linear subspace X spanned by $\{x_1, \ldots, x_m\}$ is a Hausdorff topological space in the relative topology, and Theorem 3 gives the conclusion. ∎

5 Corollary *A finite-dimensional subspace of a topological vector space is closed.*

6 Example (Corollary 4 fails without Hausdorff property) Let $X = \boldsymbol{R}^2$ under the semi-metric $d\big((x,y),(x',y')\big) = |x - x'|$. (This topology is not Hausdorff.) Then X is a topological vector space. Let $x_1 = (1,0)$ and $x_2 = (0,1)$ be the unit coordinate vectors. Then $\frac{1}{m}x_1 + 0x_2 = (1/m, 0) \to (0,1) = 0x_1 + 1x_2$, (since $d\big((1/m,0),(0,1)\big) = 1/m$, but the second coordinates do not converge $(0 \not\to 1)$. □

Appendices

The next sections cover some of the prerequisites for the results above, just in case some of the terms are new.

A Topological spaces

You should know that the collection of open subsets of $\boldsymbol{R}^{\mathrm{m}}$ is closed under finite intersections and arbitrary unions. Use that as the motivation for the following definition.

7 Definition *A* ***topology*** *τ on a nonempty set X is a family of subsets of X, called* ***open sets*** *satisfying*

1. *$\varnothing \in \tau$ and $X \in \tau$.*
2. *The family τ is closed under finite intersections. That is, if $U_1, \ldots, U_m$ belong to τ, then $\bigcap_{i=1}^m U_i$ belongs to τ.*
3. *The family τ is closed under arbitrary unions. That is, if U_α, $\alpha \in A$, belong to τ, then $\bigcup_{\alpha\in A} U_\alpha$ belongs to τ.*

The pair (X, τ) is a ***topological space****.*

The topology τ is a ***Hausdorff*** *topology if for every two distinct points x, y in X there are disjoint open sets U, V with $x \in U$ and $y \in V$.*

The set A is a ***neighborhood*** *of x if there is an open set U satisfying $x \in U \subset A$.*

A set is ***closed*** *if its complement is open.*

8 Lemma *A set is open if and only it is a neighborhood of each of it points.*

Proof: Clearly an open set is a neighborhood of each of its points. So assume the set G is a neighborhood of each of it points. That is, for each $x \in G$ there is an open set U_x satisfying $x \in U_x \subset G$. Then $G = \bigcup_{x \in G} U_x$ is open, being a union open sets. ∎

The collection of open sets in $\boldsymbol{R}^{\mathrm{m}}$ is a Hausdorff topology. A property of X that can be expressed in terms of its topology is called a topological property.

9 Definition *Let X and Y be topological spaces and let $f\colon X \to Y$. Then f is* ***continuous*** *if the inverse image of open sets are open. That is, if U is an open subset of Y, then $f^{-1}(U)$ is an open subset of X.*

This corresponds to the usual ε-δ definition of continuity that you are familiar with.

10 Definition *A family $\mathcal{G}$ of open sets is a* ***base*** *(or basis) for the topology τ if every open set in τ is a union of sets from $\mathcal{G}$. A* ***neighborhood base at x*** *is a collection $\mathcal{N}$ of neighborhoods of x such that for every neighborhood G of x there is a neighborhood U of x belong to $\mathcal{N}$ satisfying $x \in U \subset G$.*

In a metric space, the collection of open balls $\{B_\varepsilon(x) : \varepsilon > 0,\ x \in X\}$ is base for the metric topology, and $\{B_{1/n}(x) : n > 0\}$ is a neighborhood base at x.

Given a nonempty family $\mathcal{A}$ of subsets of X there is a smallest topology $\tau_{\mathcal{A}}$ on X that includes $\mathcal{A}$, called the **topology generated by $\mathcal{A}$**. It consists of arbitrary unions of finite intersections of members of $\mathcal{A}$. If $\mathcal{A}$ is closed under finite intersections, then $\mathcal{A}$ is a base for the topology $\tau_{\mathcal{A}}$.

11 Definition *If X and Y are topological spaces, the collection sets of the form $U \times V$, where U is an open set in X and V is an open set in Y, is closed under finite intersections, so it is a base for the topology it generates on $X \times Y$, called the* ***product topology****.*

12 Definition *Let X and Y be topological spaces. A function $f\colon X \to Y$ is a* ***homeomorphism*** *if it is a bijection (one-to-one and onto), is continuous, and its inverse is continuous.*

If f is homeomorphism $U \leftrightarrow f(U)$ is a one-to-one correspondence between the topologies of X and Y. Thus X and Y have the same topological properties. They can in effect be viewed as the same topological space, where f simply renames the points.

13 Definition *A set K in a topological space X is* ***compact*** *if for every family $\mathcal{G}$ of open sets satisfying $K \subset \cup\mathcal{G}$ (an* ***open cover*** *of K), there is a finite subfamily $\{G_1, \ldots, G_k\} \subset \mathcal{G}$ with $K \subset \bigcup_{i=1}^{k} G_i$ (a* ***finite subcover*** *of K).*

The following is well known.

14 Heine–Borel–Lebesgue Theorem *A subset of $\boldsymbol{R}^{\mathrm{m}}$ is compact if and only if it is closed and bounded.*

15 Lemma *A closed subset of a compact set is compact.*

Proof: Let K be compact and $F \subset K$ be closed. Let $\mathcal{G}$ be an open cover of F. Then $\mathcal{G} \cup \{F^c\}$ is an open cover of K. Let $\{G_1, \dots, G_k, F^c\}$ be a finite subcover of K. Then $\{G_1, \dots, G_k\}$ is a finite subcover of F. ∎

16 Lemma *A compact subset of a Hausdorff space is closed.*

Proof: Let K be compact, and let $x \notin K$. Then by the Hausdorff property, for each $y \in K$ there are disjoint open sets U_y and V_y with $y \in U_y$ and $x \in V_y$. By compactness there are $y_1, \dots, y_k$ with $K \subset \bigcup_{i=1}^k U_{y_i} = U$. Then $V = \bigcap_{i=1}^k V_{y_i}$ is an open set satisfying $x \in V \subset U^c \subset K^c$. That is, K^c is a neighborhood of x. Since x is an arbitrary member of K^c, we see that K^c is open (Lemma 8), so K is closed. ∎

17 Lemma *Let $f \colon X \to Y$ be continuous. If K is a compact subset of X, then $f(K)$ is a compact subset of Y.*

Proof: Let $\mathcal{G}$ be an open cover of $f(K)$. Then $\{f^{-1}(G) : G \in \mathcal{G}\}$ is an open cover of K. Let $\{f^{-1}(G_1), \dots, f^{-1}(G_k)\}$ be a finite subcover of K. Then $\{G_1, \dots, G_k\}$ is a finite subcover of $f(K)$. ∎

18 Definition (Relative topology) *If (X, τ) is a topological space and $A \subset X$, then (A, τ_A) is a topological space with its* ***relative topology****, where $\tau_A = \{G \cap A : G \in \tau\}$.*

Not that if τ is a Hausdorff topology, then τ_A is also a Hausdorff topology.

19 Lemma *If (X, τ) is a topological space and $K \subset A \subset X$, then K is a compact subset of (A, τ_A) if and only if it is a compact subset of (X, τ).*

Proof: Assume K is a compact subset of (X, τ). Let $\mathcal{G}$ be a τ_A-open cover of K in A. For each $G \in \mathcal{G}$ there is some $U_G \in \tau$ with $G = U_G \cap A$. Then $\{U_G : G \in \mathcal{G}\}$ is a τ-open cover of K in X, so it has a finite subcover $U_{G_1}, \dots, U_{G_k}$. But then $G_1, \dots, G_k$ is a finite subcover of K in A.

The converse is similar. ∎

20 Lemma *Let $f \colon X \to Y$ be one-to-one and continuous, where Y is a Hausdorff space and X is compact. The $f \colon X \to f(X)$ is a homeomorphism, where $f(X)$ has its relative topology as a subset of Y.*

Proof: We need to show that the function $f^{-1} \colon f(X) \to X$ is continuous. So let G be any open subset of X. We must show that $(f^{-1})^{-1}(G) = f(G)$ is open in $f(X)$. Now G^c is a closed subset of X, and thus compact. Therefore $f(G^c)$ is compact, and since Y is Hausdorff, so is $f(X)$, so $f(G^c)$ is a closed subset of Y. Now $f(X) \cap f(G^c)^c = f(G)$, so $f(G)$ is open in $f(X)$. ∎

B Topological vector spaces

21 Definition *A (real)* ***topological vector space*** *is a vector space X together with a topology τ where τ has the property that the mappings scalar multiplication and vector addition are continuous functions. That is, the mappings*

$$(\alpha, x) \mapsto \alpha x$$

from $\boldsymbol{R} \times X$ *to* X *and*

$$(x, y) \mapsto x + y$$

from $X \times X$ *to* X *are continuous. (Where, of course,* $\boldsymbol{R}$ *has its usual topology, and* $\boldsymbol{R} \times X$ *and* $X \times X$ *have their product topologies.)*

For a detailed discussion of topological vector spaces, see chapter five of the Hitchhiker's Guide [1]. But here are some of the results we will need.

A set A in the vector space X is **circled** if for each $x \in A$ the line segment joining the points x and $-x$ lies in A.

Let X be a topological vector space. For each fixed scalar $\alpha \neq 0$ the mapping $x \mapsto \alpha x$ has an inverse $x \mapsto \frac{1}{\alpha}x$. Each of these maps is continuous, so each is a homeomorphism. Thus αV is a neighborhood of zero whenever V is and $\alpha \neq 0$. Now if V is a neighborhood of zero, then the continuity of the function $(\alpha, x) \mapsto \alpha x$ at $(0, 0)$ guarantees the existence of a neighborhood W at zero and some $\alpha_0 > 0$ such that $x \in W$ and $|\alpha| \leq \alpha_0$ imply $\alpha x \in V$. Thus, if $U = \bigcup_{|\alpha| \leq \alpha_0} \alpha W$, then U is a neighborhood of zero, $U \subset V$, and U is circled. Therefore there is a neighborhood base of circled sets at zero.

For each vector y the maps $x \mapsto x + y$ and $x \mapsto x - y$ are continuous and mutual inverses, and so homeorphisms. Thus a set G is open if and only its translation $y + G$. Therefore the topology on X is completely determined by its neighborhoods of zero and a linear mapping between topological vector spaces is continuous if and only if it is continuous at zero.

References

[1] Aliprantis, C. D. and K. C. Border. 2006. *Infinite dimensional analysis: A hitchhiker's guide*, 3d. ed. Berlin: Springer–Verlag.

[2] Franklin, J. 2002. *Methods of mathematical economics: Linear and nonlinear programming, fixed point theorems*. Number 37 in Classics in Applied Mathematics. Philadelphia: SIAM. Corrected reprint of the 1980 edition published by Springer–Verlag.