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On continued fractions

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A fundamental metric space is the **B**aire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ of functions from $\mathbb{N}$ into $\mathbb{N}$ (or sequences of natural numbers), endowed with its product topology. Since the discrete metric on $\mathbb{N}$ is complete, $\mathbb{N}$ is a Polish space. Thus $\mathcal{N}$ is Polish too.

Surprisingly $\mathcal{N}$ is homeomorphic to the space $\mathcal{I}$ of irrational numbers in $(0, 1)$ (with their usual metric topology). The "classical" proof uses some results from the theory of infinite continued fractions, which is not widely taught these days. A proof along these lines may be found in Bertsekas and Shreve [1, Proposition 7.5, pp. 109–112]. For an elementary introduction to continued fractions, see Olds [2].

Theorem 1 (The space of irrationals) *The Baire space $\mathcal{N}$ is homeomorphic to the space $\mathcal{I}$ of irrationals in $(0, 1)$.*

Proof via continued fractions: We start with an algorithm for mapping $\mathcal{I}$ into $\mathcal{N}$. Given $x \in \mathcal{I}$, set $x_0 = x$. Note that $\frac{1}{x_0} > 1$. Recursively define x_m and n_m by $\frac{1}{x_m} = n_{m+1} + x_{m+1}$, where $n_{m+1} = [\frac{1}{x_m}]$, the largest integer less than or equal to $\frac{1}{x_m}$, and $x_{m+1} \in (0, 1)$ is irrational. Notice that this process does not terminate. Then $x = x_0 = \frac{1}{n_1+x_1}$, $x_1 = \frac{1}{n_2+x_2}$, etc. Consequently, $x = \frac{1}{n_1+x_1} = \frac{1}{n_1+\frac{1}{n_2+x_2}}$, etc. We can express this sequence of equalities formally as an infinite **continued fraction**:

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}}} \tag{1}$$

This algorithm assigns to each irrational x in $(0, 1)$ a unique sequence $f(x) = (n_1, n_2, \ldots)$ belonging to $\mathcal{N}$. It is easy to see that if a sequence

$\{x_m\}$ of irrationals converges to an irrational x, then $f(x_m)$ converges to $f(x)$ pointwise in $\mathcal{N}$.

To see that f is one-to-one and surjective, we now describe an algorithm that inverts the above process. This algorithm thus gives a meaning to the infinite continued fraction (1) above.

Given a point $(n_1, n_2, \ldots) \in \mathcal{N}$, the numbers

$$c_1 = \frac{1}{n_1}, \quad c_2 = \frac{1}{n_1 + \frac{1}{n_2}}, \quad c_3 = \frac{1}{n_1 + \cfrac{1}{n_2 + \frac{1}{n_3}}}, \quad \text{etc.,}$$

are called the **convergents** of the continued fraction (1). It is not hard to see that

$$0 < c_2 < c_4 < c_6 < \cdots \quad \cdots < c_5 < c_3 < c_1 < 1, \tag{2}$$

Therefore we need only show that $c_k - c_{k-1} \xrightarrow[k\to\infty]{} 0$.

To this end, for each c_k associate the set of numbers $\{a_1^k, \ldots, a_k^k, a_{k+1}^k\}$ via $a_{k+1}^k = 0$, $a_k^k = \frac{1}{n_k}$, and $a_s^k = \frac{1}{n_s + a_{s+1}^k}$ for $1 \leq s \leq k-1$. For $k \geq 2$, we have

$$\begin{aligned} |c_k - c_{k-1}| &= \left| \tfrac{1}{n_1 + a_2^k} - \tfrac{1}{n_1 + a_2^{k-1}} \right| = \tfrac{|a_2^{k-1} - a_2^k|}{(n_1 + a_2^k)(n_1 + a_2^{k-1})} \\ &= \frac{|a_3^{k-1} - a_3^k|}{(n_1 + a_2^k)(n_1 + a_2^{k-1})(n_2 + a_3^k)(n_2 + a_3^{k-1})} \\ &= \tfrac{1}{n_k} \prod_{s=1}^{k-1} \frac{1}{(n_s + a_{s+1}^k)(n_s + a_{s+1}^{k-1})}. \end{aligned}$$

If $n_i \geq 2$ for an infinite number of indices i, then clearly $c_k - c_{k-1} \xrightarrow[k\to\infty]{} 0$. On the other hand, of $n_i = 1$ for all $i \geq m$, then for all $k \geq m$, we see that $a_k^k = 1$ and $a_s^k = \frac{1}{1 + a_{s+1}^k} > \frac{1}{2}$ for $m \leq s \leq k-1$. So $c_k - c_{k-1} \xrightarrow[k\to\infty]{} 0$ in this case also.

Therefore there is some $x \in \boldsymbol{R}$ with $c_k \xrightarrow[k\to\infty]{} x$. We claim that for this x the first algorithm yields $f(x) = (n_1, n_2, \ldots)$. To see this, note that if x is the specified limit, then (2) implies that $\frac{1}{n_1 + \frac{1}{n_2}} < x < \frac{1}{n_1}$. From this we see that if we write $x = \frac{1}{n_1 + x_1}$, then it must be the case that $0 < x_1 < \frac{1}{n_2} \leqslant 1$, so that n_1 is the first term generated by our first algorithm. Proceeding inductively, you can see that $f(x) = (n_1, n_2, \ldots)$, as desired.

There is one point left. How do we know that the x generated by the second algorithm is irrational? The answer is that if x is rational, eventually the first algorithm stops, so it cannot generate an infinite sequence of natural numbers. (Remember, zero is not a natural number.) To see this, recall the familiar Euclidean algorithm

$$\tfrac{q}{p} = n_1 + \tfrac{r_1}{p},\ \ \tfrac{p}{r_1} = n_2 + \tfrac{r_2}{r_1},\ \ \tfrac{r_1}{r_2} = n_3 + \tfrac{r_3}{r_2}, \dots ,$$

where $r_i \geq 0$ and $r_i < r_{i-1}$. Hence, if x is rational, r_i must eventually be zero, so the process stops after a finite number of steps. Thus x is irrational.

Finally, we leave the verification that f is indeed a homeomorphism as an exercise. ∎

References

[1] D. P. Bertsekas and S. E. Shreve. 1978. *Stochastic optimal control: The discrete time case.* Number 139 in Mathematics in Science and Engineering. New York: Academic Press. hdl.handle.net/1721.1/4852

[2] C. D. Olds. 1963. *Continued fractions.* New York: Random House.