Caltech Division of the Humanities and Social Sciences

Preliminary notes on auction design

kcb

Revised Winter 2008

This note exposits a simplified version of Myerson's [8] paper on revenue-maximizing auction design for independent private values.

1 Auction mechanisms

A seller has a single object to sell to one of N bidders. The seller has no use for the object himself. Bidders are assumed to be identical except for their **type**. A bidder's type is a represented by a real number θ in the interval

$$\Theta = [\underline{\theta}, \overline{\theta}] \subset \boldsymbol{R}_+.$$

A typical element in Θ^N is denoted $\boldsymbol{\theta}$.

A **direct auction mechanism** determines the allocation of the object and the payments to the bidders based on the vector of the bidders' types. That is, it is a pair of functions

$$\boldsymbol{q}\colon \Theta^N \to [0,1]^N, \quad \boldsymbol{t}\colon \Theta^N \to \boldsymbol{R}^N.$$

The **allocation rule $\boldsymbol{q}$** determines the probability $\boldsymbol{q}_i(\boldsymbol{\theta})$ that bidder i is awarded ("wins") the object when the vector of types is $\boldsymbol{\theta}$. We allow for random assignment, in part to have symmetric tie-breaking rules. The function $\boldsymbol{q}$ must satisfy

$$\boldsymbol{q}_i(\boldsymbol{\theta}) \geqslant 0, \qquad \boldsymbol{\theta} \in \Theta^N,\ i = 1, \dots, N$$

and

$$\sum_{i=1}^{N} \boldsymbol{q}_i(\boldsymbol{\theta}) \leqslant 1, \qquad \boldsymbol{\theta} \in \Theta^N.$$

The **transfer rule** determines the payment $t_i(\boldsymbol{\theta})$ from bidder i to the seller when the vector of types is $\boldsymbol{\theta}$. Note that a payment may occur even if the bidder does not win the object. Such would be the case in an auction in which there was a participation fee.[1]

[1] Most auctions found in the wild do not use transfer rules that strictly act this way. The problem is with breaking ties. In the event of a two-way tie, there may be a coin toss and the object is awarded to the winner of the toss. This aspect is taken care of by allowing $\boldsymbol{q}_i$ to be a probability. But then the payment is usually conditioned on the outcome of the coin toss. That is, the winner pays and the loser doesn't. Our formulation does not allow us to condition on the outcome of the coin toss. Instead we would require that each pay half the winning price. In terms of expectation it doesn't matter, and changing the transfers to depend on a randomizing device is not hard to incorporate, but it adds one more piece of notation to cloud the argument.

2 Preferences

The (expected) **utility** of a bidder from participating in the auction is given by

$$u(q, t; \theta) = \theta q - t,$$

where θ is the bidder's type, q is the probability that he wins the object at auction, and t is a monetary payment to the seller. We have normalized utility so if the bidder does not participate (which we can view as $q = 0$ and $t = 0$) his utility is 0. Note that in this formulation the type is the monetary value of the object. This utility function embodies two kinds of **risk neutrality**—the payoff is linear in the probability of winning the object, and linear in transfers. Thus there is no point in randomizing transfers, they can be replaced by expected values.

We assume that the types are independently and identically distributed on Θ with density f and cumulative distribution function F. For convenience, assume that f is continuous and $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. This guarantees that F is strictly increasing and continuously differentiable.

The seller is also risk neutral and wants to choose $(\boldsymbol{q}, \boldsymbol{t})$ to maximize his **expected revenue**

$$\int_{\Theta^N} \sum_{i=1}^{N} \boldsymbol{t}_i(\boldsymbol{\theta})\, dF^N(\boldsymbol{\theta}).$$

We assume the seller cannot coerce payments from the bidders so he faces a **non-coercion constraint** that limits the $\boldsymbol{t}_i$.

2.1 Differences from Myerson [8]

This paper is not as general as Myerson's in the following regards.

- I assume that bidders' values are independently and identically distributed, whereas he assumes only that they are independent. This means I don't have to carry around as many subscript $_i$s as he does, but other than that it doesn't simplify things very much. I'm tempted to go back and rewrite the notes without identical distributions.

- I assume that the seller's valuation is zero, where Myerson allows it to be θ_0. Later on I introduce a type θ° defined by $\tau(\theta^\circ) = 0$. I should replace this by $\tau(\theta^\circ) = \theta_0$. If you understand my case, you will have no difficulty dealing with the case $\theta_0 \neq 0$. Although do be aware that my θ° and his θ_0 are not the same thing.

- Myerson allows for additive "revision effects," in which others' valuations affect your own. This takes us out of the strictly independent private values model, but the main difference in the analysis is that I can dispense with a term that otherwise plays no important role. The general common values model does not in general have additive revision effects, so Myerson's model is perhaps not as general as it may seem.

3 Incentive compatibility

The **revelation principle** asserts that it is enough to look at incentive compatible direct mechanisms. That is, those for which

$$\int_{\Theta^{N-1}} \theta_i \boldsymbol{q}_i(\boldsymbol{\theta}) - \boldsymbol{t}_i(\boldsymbol{\theta})\, dF^{N-1}(\boldsymbol{\theta}_{-i}) \geqslant \int_{\Theta^{N-1}} \theta_i \boldsymbol{q}_i(\boldsymbol{\theta}_{-i}, \theta') - \boldsymbol{t}_i(\boldsymbol{\theta}_{-i}, \theta')\, dF^{N-1}(\boldsymbol{\theta}_{-i}), \qquad \textbf{(IC)}$$

for all $i = 1, \ldots, N$, $\boldsymbol{\theta} \in \Theta^N$, and $\theta' \in \Theta$. In this section we examine properties of incentive compatible direct mechanisms without worrying about revenue-maximization.

1 Definition (Reduced form) *Given an auction mechanism* $(\boldsymbol{q}, \boldsymbol{t})$*, define the **reduced form allocation** rule by*

$$\boldsymbol{Q}_i(\theta) = \int_{\Theta^{N-1}} \boldsymbol{q}_i(\boldsymbol{\theta}_{-i}, \theta)\, dF^{N-1}(\boldsymbol{\theta}_{-i}) \tag{1}$$

*and **reduced form transfer** rule by*

$$\boldsymbol{T}_i(\theta) = \int_{\Theta^{N-1}} \boldsymbol{t}_i(\boldsymbol{\theta}_{-i}, \theta)\, dF^{N-1}(\boldsymbol{\theta}_{-i}). \tag{2}$$

Harris and Raviv [2] show that it suffices to consider the case where all the $\boldsymbol{Q}_i$ and $\boldsymbol{T}_i$ are identical.[2] So from now on, in order to simplify notation, we may assume that $\boldsymbol{Q}_i$ and $\boldsymbol{T}_i$ are independent of i.

Then (**IC**) can be rewritten in terms of the reduced form as

$$\theta \boldsymbol{Q}_i(\theta) - \boldsymbol{T}_i(\theta) \geqslant \theta \boldsymbol{Q}_i(\theta') - \boldsymbol{T}_i(\theta'), \qquad \begin{matrix} i = 1, \ldots, N \\ \theta, \theta' \in \Theta. \end{matrix} \qquad \textbf{(IC}'\textbf{)}$$

2 Definition (Indirect utility) *Given an incentive compatible auction mechanism, define the **indirect utility** or **optimal value function*** V_i *for bidder* i*,*[3] *by*

$$V_i(\theta) = \theta \boldsymbol{Q}_i(\theta) - \boldsymbol{T}_i(\theta). \tag{3}$$

Then (**IC′**) can be rewritten

$$V_i(\theta) \geqslant \theta \boldsymbol{Q}_i(\theta') - \boldsymbol{T}_i(\theta'), \qquad \begin{matrix} i = 1, \ldots, N \\ \theta, \theta' \in \Theta. \end{matrix} \qquad \textbf{(IC}''\textbf{)}$$

But (3) tells us the following.

---

[2] The argument runs something like this. If the functions are not identical, symmetrize the auction by randomly assign the players numbers from 1 to N. Then the incentive constraints still hold and the seller is indifferent. This amounts to replacing each $\boldsymbol{Q}_i$ and $\boldsymbol{T}_i$ with an equally weighted convex combination.

[3] It is the optimal value function for the parametrized problem

$$\underset{\theta'}{\text{maximize}}\ \theta \boldsymbol{Q}_i(\theta') - \boldsymbol{T}_i(\theta').$$

If $\boldsymbol{Q}_i$ is continuous, standard versions of the Envelope Theorem tell us that $V_i'(\theta) = \boldsymbol{Q}_i(\theta)$. But we don't know that $\boldsymbol{Q}_i$ is continuous, but Lemma 5 below gives us something just as good.

3 Lemma *If* $(\boldsymbol{Q}, \boldsymbol{T})$ *is the reduced form of an incentive compatible auction with indirect utility* V*, then*

$$\boxed{\boldsymbol{T}_i(\theta) = \theta \boldsymbol{Q}_i(\theta) - V_i(\theta),} \tag{4}$$

so $(\mathbf{IC''})$ *can be rewritten as*

$$V_i(\theta) \geqslant \theta \boldsymbol{Q}_i(\theta') + V_i(\theta') - \theta' \boldsymbol{Q}_i(\theta'), \qquad \begin{matrix} i = 1, \dots, N \\ \theta, \theta' \in \Theta, \end{matrix}$$

or

$$V_i(\theta) \geqslant V_i(\theta') + \boldsymbol{Q}_i(\theta')(\theta - \theta'), \qquad \begin{matrix} i = 1, \dots, N \\ \theta, \theta' \in \Theta. \end{matrix} \tag{$\mathbf{IC'''}$}$$

This is Myerson's equation (4.6). It asserts that $\boldsymbol{Q}_i(\theta')$ satisfies the **subgradient inequality** for V_i at θ'. In fact, we have the following.

4 Lemma *If* V *is the indirect utility of an incentive compatible auction mechanism, then*

$$V(\theta) = \sup_{\theta' \in \Theta} g_{\theta'}(\theta) = g_\theta(\theta),$$

where $g_{\theta'}$ *is the affine function*

$$g_{\theta'}(\theta) = V(\theta') + \boldsymbol{Q}(\theta')(\theta - \theta').$$

Since each function $g_{\theta'}$ is affine as a function of θ, some standard results from convex analysis [9] yield the following lemma.

5 Lemma *If* V *is the indirect utility of an incentive compatible auction mechanism, then* V *is convex and* $\boldsymbol{Q}(\theta) \in \partial V(\theta)$ *for each* θ*. Consequently* $\boldsymbol{Q}$ *is nondecreasing, and*

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \boldsymbol{Q}(s)\, ds, \qquad \theta \in \Theta. \tag{5}$$

The constraints that $\boldsymbol{Q}(\theta) \in \partial V(\theta)$ reduce to $\boldsymbol{Q}(\theta) = V'(\theta)$ if V is differentiable at θ, and are called the **local incentive constraints**. The constraint that V is convex embodies the **global incentive constraints**. It is also well known that if $\boldsymbol{Q}$ is nondecreasing, then (5) implies that V is convex.

3.1 Fubini's Theorem and "virtual types"

By Lemma 5, for an incentive-compatible auction mechanism, we have

$$\begin{aligned}\int_\Theta \Big[\theta \boldsymbol{Q}(\theta) - V(\theta)\Big] f(\theta)\,d\theta &= \int_{\underline{\theta}}^{\overline{\theta}} \left[\theta \boldsymbol{Q}(\theta) - \left(V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \boldsymbol{Q}(s)\,ds\right)\right] f(\theta)\,d\theta \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \theta \boldsymbol{Q}(\theta) f(\theta)\,d\theta - \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} \boldsymbol{Q}(s) f(\theta)\,ds\,d\theta - V(\underline{\theta}).\end{aligned} \tag{6}$$

Let us concentrate on the middle term for a minute. First we shall eliminate θ from the limit of integration by using an indicator function.

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} \boldsymbol{Q}(s) f(\theta)\,ds\,d\theta \quad = \quad \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \mathbf{1}_{[s\leqslant\theta]} \boldsymbol{Q}(s) f(\theta)\,ds\,d\theta$$

where

$$\mathbf{1}_{[s\leqslant\theta]} = \begin{cases} 1 & s \leqslant \theta \\ 0 & \text{otherwise.} \end{cases}$$

Now that we have a double integral over a fixed set, we can use Fubini's Theorem to simplify the expression.

$$\begin{aligned}\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \mathbf{1}_{[s\leqslant\theta]} \boldsymbol{Q}(s) f(\theta)\,ds\,d\theta &= \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \mathbf{1}_{[\theta\geqslant s]} f(\theta)\,d\theta\right) \boldsymbol{Q}(s)\,ds \\ &= \int_{\underline{\theta}}^{\overline{\theta}} (1 - F(s)) \boldsymbol{Q}(s)\,ds \\ &= \int_{\underline{\theta}}^{\overline{\theta}} (1 - F(\theta)) \boldsymbol{Q}(\theta)\,d\theta,\end{aligned}$$

where the last equality is just a change of dummy variable. That is,

$$\int_\Theta V(\theta) f(\theta)\,d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} \boldsymbol{Q}(s) f(\theta)\,ds\,d\theta = \int_\Theta (1 - F(\theta)) \boldsymbol{Q}(\theta)\,d\theta.$$

This means we can rewrite (6) as

$$\int_\Theta \Big[\theta \boldsymbol{Q}(\theta) - V(\theta)\Big] f(\theta)\,d\theta = \int_\Theta \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) \boldsymbol{Q}(\theta) f(\theta)\,d\theta - V(\underline{\theta}). \tag{7}$$

Myerson defines the **virtual type** τ of type θ by

$$\boxed{\tau(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}.}$$

It follows from (4) and (7) that the seller's expected revenue from an incentive-compatible direct auction can be written

$$\int_\Theta T(\theta) f(\theta)\, d\theta = \int_\Theta \tau(\theta)\boldsymbol{Q}(\theta) f(\theta)\, d\theta - V(\underline{\theta}). \tag{8}$$

Note that this does *not* imply that $\boldsymbol{T}(\theta) = \tau(\theta)\boldsymbol{Q}(\theta)$ even when $V(\underline{\theta}) = 0$, and in general they are different.

As an aside we mention the following fact.

6 Remark

$$\int_\Theta \tau(\theta) f(\theta)\, d\theta = 0.$$

Consequently, if τ is strictly increasing, then $\tau(\underline{\theta}) < 0$ and $\tau(\overline{\theta}) > 0$.

Proof: By definition,

$$\int_{\underline{\theta}}^{\overline{\theta}} \tau(\theta) f(\theta)\, d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)\, d\theta - \int_{\underline{\theta}}^{\overline{\theta}} 1 - F(\theta)\, d\theta,$$

and each of the last two integrals is just the expected value of θ. ∎

4 The revenue-maximization problem

Thus the seller's auction design problem can be formulated as an optimization problem.

Choose $V, \boldsymbol{Q}$ to

$$\text{maximize} \quad N \int_\Theta \Big[\theta \boldsymbol{Q}(\theta) - V(\theta)\Big] f(\theta)\, d\theta$$

subject to the following constraints: $(\boldsymbol{P''})$

1. $V(\theta) \geqslant V(\theta') + \boldsymbol{Q}(\theta')(\theta - \theta'), \qquad \theta, \theta' \in \Theta.$
2. $\boldsymbol{Q}$ is a reduced form of some symmetric $\boldsymbol{q}$.
3. Participation.

Matthews [5] shows that the reduced form constraint can be rewritten as

$$N \int_\theta^{\overline{\theta}} \boldsymbol{Q}(s) f(s)\, ds \leqslant 1 - F(\theta)^N, \qquad \text{for all } \theta \in \Theta.$$

We have normalized utility so if the bidder does not participate (which we can view as $q = 0$ and $t = 0$) his utility is 0, so the non-coercion constraint is

$$V(\theta) \geqslant 0, \qquad \theta \in \Theta.$$

But since $\boldsymbol{Q}(\theta) \in [0,1]$, it follows that V is nondecreasing, so the non-coercion constraint reduces to

$$V(\underline{\theta}) \geqslant 0.$$

which will bind at a revenue maximum.

Finally we can write the seller's problem as

Choose $V, \boldsymbol{Q}$, to

$$\text{maximize} \quad N \int_{\Theta} \Big[\theta \boldsymbol{Q}(\theta) - V(\theta)\Big] f(\theta)\, d\theta$$

subject to the following constraints: $(\boldsymbol{P'})$

1. V is convex.
2. $\boldsymbol{Q}(\theta) \in \partial V(\theta)$ for all $\theta \in \Theta$.
3. $V(\theta) \geqslant 0$, for all $\theta \in \Theta$.
4. $N \int_{\theta}^{\bar{\theta}} \boldsymbol{Q}(s) f(s)\, ds \leqslant 1 - F(\theta)^N$, for all $\theta \in \Theta$.

4.1 Final reformulation

Expanding $\boldsymbol{Q}$ in (7) we can rewrite the objective as

$$\begin{aligned} & N \int_{\underline{\theta}}^{\bar{\theta}} \tau(\theta_i) \boldsymbol{Q}(\theta_i) f(\theta_i)\, d\theta_i \\ &= \sum_{i=1}^{N} \int_{\underline{\theta}}^{\bar{\theta}} \tau(\boldsymbol{\theta}_i) f(\boldsymbol{\theta}_i) \left(\int_{\Theta^{N-1}} \boldsymbol{q}_i(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i)\, dF^{N-1}(\boldsymbol{\theta}_{-i}) \right) d\boldsymbol{\theta}_i \\ &= \int_{\Theta^N} \left\{ \sum_{i=1}^{N} \tau(\boldsymbol{\theta}_i) \boldsymbol{q}_i(\boldsymbol{\theta}) \right\} dF^N(\boldsymbol{\theta}). \end{aligned}$$

We are now in a position to write the problem in a halfway tractable form.

Choose $\boldsymbol{q} \colon \Theta^N \to [0,1]^N$ to

$$\text{maximize} \quad \int_{\Theta^N} \left\{ \sum_{i=1}^{N} \tau(\boldsymbol{\theta}_i) \boldsymbol{q}_i(\boldsymbol{\theta}) \right\} dF^N(\boldsymbol{\theta})$$

subject to the following constraints: $(\boldsymbol{P})$

1. $\boldsymbol{q}_i(\boldsymbol{\theta}) \geqslant 0$ for all $\boldsymbol{\theta} \in \Theta^N$,
2. $\sum_{i=1}^{N} \boldsymbol{q}_i(\boldsymbol{\theta}) = 1$ for all $\boldsymbol{\theta} \in \Theta^N$,
3. the function $\boldsymbol{Q}$ defined from $\boldsymbol{q}$ by (1) is nondecreasing.

4.2 The relaxed problem

For now, drop the constraint that the reduced form be nondecreasing. Then the so-called **relaxed problem** is the above problem without the global incentive constraint that $\boldsymbol{Q}$ is nondecreasing:

Choose $\boldsymbol{q}$ to

$$\text{maximize} \quad \int_{\Theta^N} \left\{ \sum_{i=1}^{N} \tau(\boldsymbol{\theta}_i) \boldsymbol{q}_i(\boldsymbol{\theta}) \right\} dF^N(\boldsymbol{\theta})$$

subject to the following constraints: $(\boldsymbol{R})$

1. $\boldsymbol{q}_i(\boldsymbol{\theta}) \geqslant 0$,
2. $\sum_{i=1}^{N} \boldsymbol{q}_i(\boldsymbol{\theta}) = 1$.

But this is easy, since the integrand can be separately maximized one point $\boldsymbol{\theta}$ at a time. The integrand is linear in $\boldsymbol{q}$, so the solution is obvious. Put all the probability on the is with the largest positive coefficient:

For each $(\boldsymbol{\theta}) \in \Theta^N$, let $M(\boldsymbol{\theta})$ be the set of i for which $\tau(\boldsymbol{\theta}_i)$ is a maximum, and let $\tau^{\max}(\boldsymbol{\theta})$ be the value of this maximum. If $\tau^{\max}(\boldsymbol{\theta}) > 0$ set $\boldsymbol{q}_i(\boldsymbol{\theta}) = 0$ for $i \notin M(\boldsymbol{\theta})$, and $\sum_{i \in I(\boldsymbol{\theta})} \boldsymbol{q}_i(\boldsymbol{\theta}) = 1$. Otherwise, set $q_i(\boldsymbol{\theta}) = 0$ for all i. The simplest way to do this is to break ties symmetrically.

7 Lemma *The allocation rule*

$$\boldsymbol{q}_i^*(\boldsymbol{\theta}) = \begin{cases} \dfrac{1}{|M(\boldsymbol{\theta})|} & \tau^{\max}(\boldsymbol{\theta}) > 0 \text{ and } i \in M(\boldsymbol{\theta}) \\ 0 & \text{otherwise} \end{cases}$$

solves the relaxed problem.

8 Proposition *The solution to the relaxed problem awards the object with probability 1 to the bidder(s) with the highest virtual type, provided this maximal virtual type is positive. Otherwise, the seller keeps the object.*

4.3 Where did $\boldsymbol{T}$ go?

The above discussion tells us how to find $\boldsymbol{q}^*$, and hence $\boldsymbol{Q}^*$, but what happened to $\boldsymbol{T}^*$? The answer is that $\boldsymbol{Q}^*$ and $V^*(\underline{\theta})$ determine everything. For (4) and (5) tell us that

$$\begin{aligned} \boldsymbol{T}^*(\theta) &= \theta \boldsymbol{Q}^*(\theta) - V^*(\theta) \\ &= \theta \boldsymbol{Q}^*(\theta) - \int_{\underline{\theta}}^{\theta} \boldsymbol{Q}^*(s)\, ds - V^*(\underline{\theta}). \end{aligned}$$

As a Corollary of (6) we get the following.

9 Revenue Equivalence Theorem *The revenue from an auction is completely determined by $\boldsymbol{q}^*$ or its reduced form $\boldsymbol{Q}^*$, and the values $V_i^*(\underline{\theta})$. In a revenue-maximizing auction, each $V_i^*(\underline{\theta}) = 0$, so $\boldsymbol{Q}^*$ determines the revenue. In particular, if two auctions result in the highest-value bidder winning, then they generate the same revenue.*

5 Myerson's regular case

Myerson [8] noted that if τ is strictly increasing in θ (which he called the **regular case**), then the reduced form $\boldsymbol{Q}^*$ induced by the solution $\boldsymbol{q}^*$ to the relaxed problem is easily seen to be increasing. (Since $\boldsymbol{q}^*$ assigns the object to the highest virtual-type, if τ is strictly increasing, then $\boldsymbol{q}^*$ assigns the object to the highest type. Thus the higher the type, the greater the probability it wins.) That is, the solution to the relaxed problem is automatically a solution to the original problem.

10 Theorem (Allocation in the revenue-maximizing auction) *Assume τ is strictly increasing in θ (Myerson's regular case). Define θ° by*

$$\boxed{\tau(\theta^\circ) = 0.}$$

The revenue-maximizing auction awards the object with probability 1 to the bidder(s) with the highest type, provided this maximal type is at least θ°. If no bidder has type at least θ°, then the seller keeps the object.

By Remark 6, in the strictly increasing case θ° exists and satisfies $\theta^\circ > \underline{\theta}$, so there is a positive probability that the object remains unsold.[4]

11 Corollary *Under the assumption of Theorem 10, a symmetric revenue-maximizing auction satisfies*

$$\boldsymbol{Q}_i(\theta) = \begin{cases} F(\theta)^{N-1} & \theta \geqslant \theta^\circ \\ 0 & \theta < \theta^\circ. \end{cases}$$

Proof: The probability that i's type is highest is just the probability that the remaining $N-1$ types are all lower. With a density we needn't worry about ties, which occur with probability zero. ∎

We can write down the formulas for the V_i and $\boldsymbol{T}_i$, but there is a more interesting question remaining.

5.1 What's the price?

Most natural auctions require bidders to pay only when they win the object. A **pricing rule** is a function $\boldsymbol{\pi}\colon \Theta^N \to \boldsymbol{R}^N$ that determines the price the winner pays and may depend on other bidders' types (as in a second-price auction). Can we rewrite the expected transfers $\boldsymbol{T}$ or $\boldsymbol{t}$ as being generated by a pricing rule $\boldsymbol{\pi}$? That is, can we find $\boldsymbol{\pi}$ satisfying

$$\boldsymbol{q}_i(\boldsymbol{\theta}) = 0 \quad \Longrightarrow \quad \boldsymbol{\pi}_i(\boldsymbol{\theta}) = 0,$$

[4]You might wonder why the seller does not sell the object to the highest value bidder anyhow, since he now knows the bidders' types and doesn't value the object himself. The answer is that if the bidders believed this to be a possibility, they would alter their bids in the first round, and the mechanism is no longer incentive compatible. We could, of course, started with a two stage mechanism and analyzed it, but then we would find the need for a third stage, etc. This leads to the literature on what is awkwardly named "renegotiation-proofness."

and

$$\boldsymbol{T}_i(\theta) = \int_{\Theta^{N-1}} \boldsymbol{\pi}_i(\boldsymbol{\theta}_{-i}, \theta)\, dF^{N-1}(\boldsymbol{\theta}_{-i}). \tag{9}$$

We shall consider only the symmetric case. The first thing to observe is that the revenue-maximizing auction wards the object to the highest-value bidder, so with probability 1 (ties have zero probability), $\boldsymbol{q}_i(\boldsymbol{\theta})$ is either 0 or 1. To simplify the description of $\boldsymbol{q}$ define

$$\boxed{\mu(\boldsymbol{\theta}_{-i}) = \max_{j \neq i} \boldsymbol{\theta}_j \vee \theta^{\circ}.}$$

Then the revenue-maximizing $\boldsymbol{q}_i$ is almost surely the indicator function

$$\boldsymbol{q}_i(\boldsymbol{\theta}) = \mathbf{1}_{[\boldsymbol{\theta}_i > \mu(\boldsymbol{\theta}_{-i})]} \text{ a.s.}$$

Now (4) and (5) imply

$$\begin{aligned}
\boldsymbol{T}_i(\theta) &= \theta \boldsymbol{Q}_i(\theta) - V_i(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \boldsymbol{Q}_i(s)\, ds \\
&= \theta \int_{\Theta^{N-1}} \boldsymbol{q}_i(\boldsymbol{\theta}_{-i}, \theta)\, dF^{N-1}(\boldsymbol{\theta}_{-i}) - \int_{\underline{\theta}}^{\theta} \left(\int_{\Theta^{N-1}} \boldsymbol{q}_i(\boldsymbol{\theta}_{-i}, s)\, dF^{N-1}(\boldsymbol{\theta}_{-i}) \right) ds
\end{aligned}$$

using the fact that $V_i(\underline{\theta}) = 0$. The second integral can be turned inside out using Fubini's Theorem.

$$\begin{aligned}
\int_{\underline{\theta}}^{\theta} \left(\int_{\Theta^{N-1}} q_i(\boldsymbol{\theta}_{-i}, s)\, dF^{N-1}(\boldsymbol{\theta}_{-i}) \right) ds &= \int_{\Theta^{N-1}} \left(\int_{\underline{\theta}}^{\theta} q_i(\boldsymbol{\theta}_{-i}, s)\, ds \right) dF^{N-1}(\boldsymbol{\theta}_{-i}) \\
&= \int_{\Theta^{N-1}} \left(\int_{\underline{\theta}}^{\theta} \mathbf{1}_{[s > \mu(\boldsymbol{\theta}_{-i})]}\, ds \right) dF^{N-1}(\boldsymbol{\theta}_{-i})
\end{aligned}$$

If $\theta > \mu(\boldsymbol{\theta}_{-i})$, then

$$\int_{\underline{\theta}}^{\theta} \mathbf{1}_{[s > \mu(\boldsymbol{\theta}_{-i})]}\, ds = \int_{\mu(\boldsymbol{\theta}_{-i})}^{\theta} 1\, ds = \theta - \mu(\boldsymbol{\theta}_{-i}),$$

and if $\theta \leqslant \mu(\boldsymbol{\theta}_{-i})$, the integral is zero. Thus we may write

$$\boldsymbol{T}_i(\theta) = \int_{\Theta^{N-1}} \theta q_i(\boldsymbol{\theta}_{-i}, \theta) - \big(\theta - \mu(\boldsymbol{\theta}_{-i})\big)^{+}\, dF^{N-1}(\boldsymbol{\theta}_{-i}).$$

But $\boldsymbol{q}_i(\boldsymbol{\theta}_{-i}, \theta) = 1$ implies $\theta \geqslant \mu(\boldsymbol{\theta}_{-i})$, in which case the integrand is $\theta - \big(\theta - \mu(\boldsymbol{\theta}_{-i})\big) = \mu(\boldsymbol{\theta}_{-i})$, and if $\theta - \mu(\boldsymbol{\theta}_{-i}) \leqslant 0$, then $\boldsymbol{q}_i(\boldsymbol{\theta}_{-i}, \theta) = 0$ almost surely, so

$$\boldsymbol{T}_i(\theta) = \int_{\Theta^{N-1}} \mu(\boldsymbol{\theta}_{-i}) \mathbf{1}_{[\boldsymbol{q}(\boldsymbol{\theta})=1]}\, dF^{N-1}(\boldsymbol{\theta}_{-i}).$$

In other words, we have found our price function,

$$\boxed{\boldsymbol{\pi}_i(\boldsymbol{\theta}) = \mu(\boldsymbol{\theta}_{-i}) \mathbf{1}_{[\boldsymbol{q}_i(\boldsymbol{\theta})=1]} = \mu(\boldsymbol{\theta}_{-i}) \boldsymbol{q}_i(\boldsymbol{\theta}) \text{ a.s.}}$$

(We probably want to modify this slightly for the zero probability event of a tie, but you get the idea.) What this says is that the highest-value bidder wins if his value is at least θ°, and then pays a price equal to the greater of θ° or the second-highest value. The value θ° acts as a **minimum bid** or **reserve price**.

12 Theorem *In Myerson's regular case, the revenue-maximizing auction is a second price auction with a reserve price.*

5.2 The reserve price

Consider the case of one bidder. In this case the revenue-maximizing auction consists of a fixed asking price, and the bidder is free to pay it or not. How should the price be set? The expected revenue from price p is just p times the probability the bidder's value is greater than p,

$$p\big(1 - F(p)\big).$$

The first-order condition for a maximum of this is

$$1 - F(p) - pf(p) = 0,$$

so

$$p - \frac{1 - F(p)}{f(p)} = \tau(p) = 0.$$ [5]

Thus the reserve price θ° is the price the seller would set if there were only one bidder. The remarkable fact is that the reserve price is independent of the number of bidders.[6]

5.3 Expected revenue

To calculate the expected revenue, there are two relevant densities, the density of the maximum

$$NF(\theta)^{N-1}f(\theta),$$

and the density of the second-highest value

$$N(N-1)F(\theta)^{N-2}\big(1 - F(\theta)\big)f(\theta).$$

The price received is the second-highest value, when it exceeds θ°, and is θ° when the second highest value is less than θ° and the maximum is greater than θ°. The expected revenue is

$$\theta^\circ NF(\theta^\circ)^{N-1}\big(1 - F(\theta)\big) + N(N-1)\int_{\theta^\circ}^{\overline{\theta}} \theta F(\theta)^{N-2}\big(1 - F(\theta)\big)f(\theta)\,d\theta.$$

[5] The function

$$\frac{f(\theta)}{1 - F(\theta)}$$

is called the **hazard rate** of distribution function F. The name comes from the following interpretation. Let t be the age of a device and let $F(t)$ be the probability that it has failed by age t. The conditional on not having failed by age T, $f(t)/\big(1 - F(T)\big)$ is the density of failure at age $t \geqslant T$. Thus $f(t)/\big(1 - F(t)\big)$ is the "instantaneous" rate of failure of a working device at age t.

[6] In the case (not discussed here) where the bidders' values are not identically distributed, the revenue-maximizing auction entails a different minimum bid for each bidder, but each satisfies $\tau_i(p) = 0$.

The bidders' indirect utility function is given by

$$V(\theta) = \begin{cases} 0 & \theta \leqslant \theta^\circ \\ \displaystyle\int_{\theta^\circ}^{\theta} F(s)^{N-1}\,ds & \theta > \theta^\circ. \end{cases}$$

6 An example

Here is a simple example. Let $\Theta = [0,1]$, and let F be the uniform distribution $F(\theta) = \theta$ and $f(\theta) = 1$ on $[0,1]$. Then

$$\tau(\theta) = 2\theta - 1,$$

which is increasing with $\theta^\circ = 1/2$. The expected revenue is

$$N(N-1)\int_{1/2}^{1} \theta^{N-2}(1-\theta)\,d\theta + N/2^{N+1}.$$

For some small values of N, Table 1 shows the expected revenue, the expected value of the highest type, and their ratio, which measures the efficiency of revenue extraction. The case of 1 bidder involves a fixed price of $\theta^\circ = 1/2$ and the object is sold with probability 1/2. You

N	Expected revenue	Expected maximum value	Extraction efficiency	Probability of sale
1	.25	.5	.5	.5
2	.291667	.666667	.4375	.75
3	.40625	.75	.541667	.875
4	.51875	.8	.648438	.9375
5	.609375	.833333	.73125	.96875
6	.677455	.857143	.790365	.984375
7	.727539	.875	.831473	.992188
8	.76454	.888889	.860107	.996094
9	.792383	.9	.880425	.998047
10	.813876	.909091	.895264	.999023

Table 1. Performance figures for the optimal auction (uniform case).

should be wary about the use of the term "efficiency" in the table. It has nothing to do with economic efficiency in the usual sense (Pareto efficiency). Since the seller has no intrinsic value, the efficient allocation is to give it to the highest-value bidder. The auction achieves this except when the highest-value bidder has value less than $\theta^\circ = 1/2$. This happens with probability 2^{-N}. The probability of sale is the proper measure of efficiency in this case.

Figure 1 plots the bidders' indirect utility functions for selected values of N. Note that bidders are worse off the more competition they face.

And for good measure let's compare $\boldsymbol{T}(\theta)$ with $\tau(\theta)\boldsymbol{Q}(\theta)$ to see that they are not equal. Recall that (8) implies that they have the same mean. Some representative values are displayed in Table 2, and graphed in Figure 2. [For $N = 1$, $\boldsymbol{Q}(\theta) = 1$ for $\theta > 1/2$, so $\tau(\theta)\boldsymbol{Q}(\theta) = (2\theta - 1)$ for

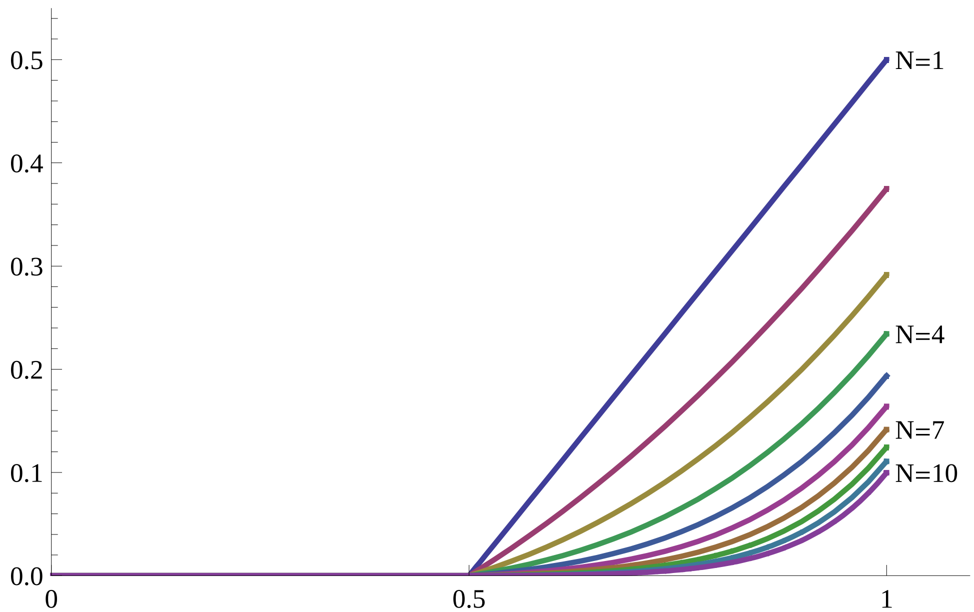

Figure 1. Indirect utility V for selected values of N (uniform case).

$\theta > 1/2$; and $\boldsymbol{T}(\theta) = \theta\boldsymbol{Q}(\theta) - \int_{1/2}^{\theta} Q(s)\,ds = 1/2$ for $\theta > 1/2$. For $N > 1$, $\boldsymbol{Q}(\theta) = \theta^{N-1}$ for $\theta > 1/2$, so $\tau(\theta)\boldsymbol{Q}(\theta) = (2\theta-1)\theta^{N-1} = 2\theta^N - \theta^{N-1}$ for $\theta > 1/2$; and $\boldsymbol{T}(\theta) = \theta\boldsymbol{Q}(\theta) - \int_{1/2}^{\theta} Q(s)\,ds = (N-1)\theta^N/N + 2^{-N}/N$ for $\theta > 1/2$.]

7 The general case

The solution to the relaxed problem is to has the following allocation rule: award the object to the bidder with the highest virtual type (provided their true value is > 0). Thus types θ with higher virtual type $\tau(\theta)$ have a higher reduced form probability $\boldsymbol{Q}(\theta)$ of winning the object. If τ is not nondecreasing, then $\boldsymbol{Q}$ is not nondecreasing and the auction mechanism is not incentive compatible. The solution in this case is to modify the allocation rule to make it nondecreasing, while not losing (too much) revenue. (Don't worry if this is not obvious to you.)

7.1 "Ironing"

We now digress to discuss a purely mathematical problem. There are (at least) two obvious increasings function derived from τ, namely, the increasing majorant and the increasing minorant. Mathematicians say that the function $\overline{g}$ **majorizes** or **dominates** g if $\overline{g}(x) \geqslant g(x)$ for all x. The **increasing majorant** $\overline{g}$ of a function g is the least increasing function that majorizes g. That is, $\overline{g}$ is nondecreasing, $\overline{g} \geqslant g$, and if h is nondecreasing and h majorizes g, then h also majorizes $\overline{g}$. The **increasing minorant** $\overline{g}$ of g is the greatest increasing function majorized by

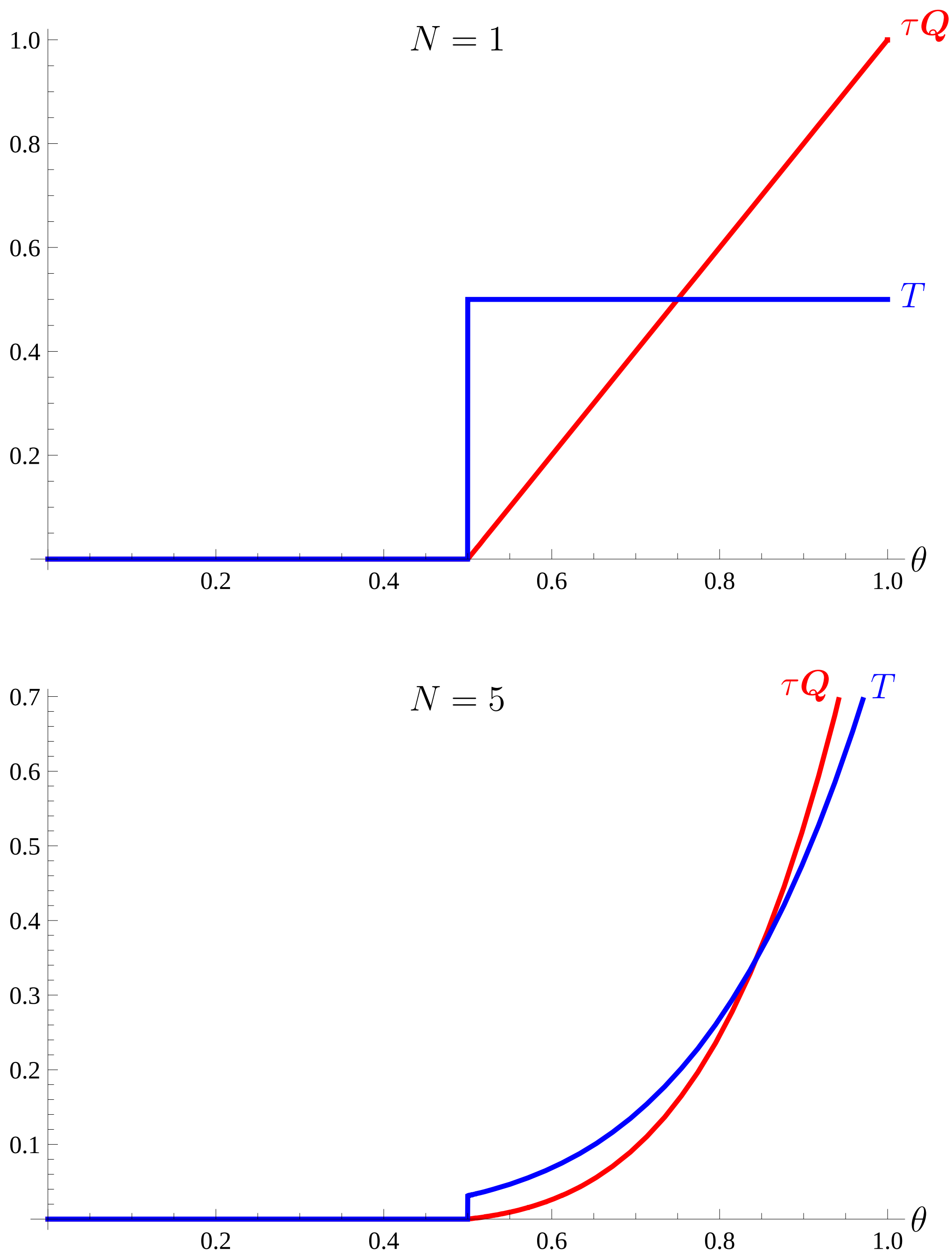

Figure 2. Reduced form payment (uniform case).

$N=1$				
θ	$T(\theta)$	$\tau(\theta)\boldsymbol{Q}(\theta)$	$Q(\theta)$	$\tau(\theta)$
.6	.5	.2	1	.2
.7	.5	.4	1	.4
.8	.5	.6	1	.6
.9	.5	.8	1	.8
1	.5	1	1	1
$N=5$				
θ	$T(\theta)$	$\tau(\theta)\boldsymbol{Q}(\theta)$	$Q(\theta)$	$\tau(\theta)$
.6	.068458	.02592	.1296	.2
.7	.140706	.09604	.2401	.4
.8	.268394	.24576	.4096	.6
.9	.478642	.52488	.6561	.8
1	.80625	1	1	1

Table 2. Reduced form (uniform case).

g.[7] See Figure 3. Neither of these two functions is the one we want.

The function $\tilde{g}$ we want lies between the increasing minorant and majorant and splits the area between it and g in half over the interval $[\underline{x}, \overline{x}]$. That is, we want

$$\int_{\underline{x}}^{\overline{x}} \tilde{g}(x)\,dx = \int_{\underline{x}}^{\overline{x}} g(x)\,dx.$$

For lack of a better term, I'll call it the **ironed** version of g (which is the terminology used by Mussa and Rosen [7] in a related situation). This is an imprecise as well as awkward way to describe $\tilde{g}$, especially if g is wiggly.

There is a theoretically simpler way to describe the ironed version. First we look a the indefinite integral G of g. The function G is convex if and only if g is nondecreasing. If G is not convex (and even if it is), its **convex envelope** $\check{G}$ is the greatest convex function it dominates. It coincides with the pointwise supremum of the affine functions that G dominates.[8] In the case of one dimension, it can be written

$$\check{G}(x) = \min\{\lambda G(y) + (1-\lambda)G(z) : 0 \leqslant \lambda \leqslant 1 \text{ and } x = \lambda y + (1-\lambda)z\}.$$

Figure 4 shows G and $\check{G}$ for the function g above. Then the ironed version of g is just the derivative (or any selection from the subdifferential) of $\check{G}$,

$$\tilde{g}(x) = \check{G}'(x).$$

[7] There are formulas for each of these functions which are not especially useful.

$$\overline{g}(x) = \sup\{f(z) : z \leqslant z\}, \qquad \underline{g}(x) = \inf\{f(z) : z \geqslant x\}.$$

These agree with g if g is nondecreasing.

[8] If G is convex, then G coincides with $\check{G}$ except possibly at the endpoints of its domain. It will coincide there too provided G is lower semicontinuous (that is, it doesn't jump up at the endpoints).

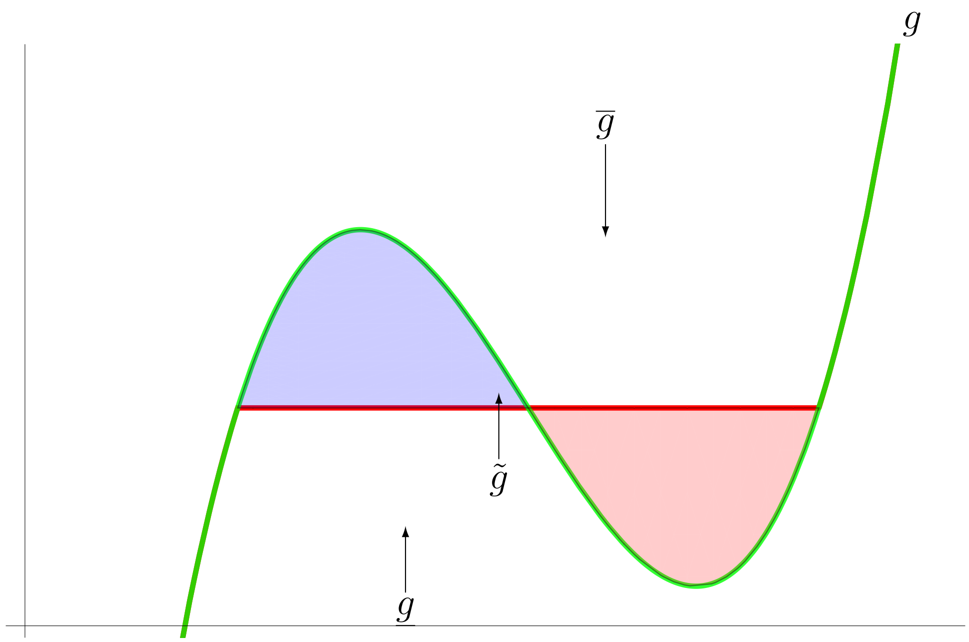

Figure 3. The increasing majorant $\overline{g}$ and minorant $\underline{g}$ of g, and the ironed version $\tilde{g}$.

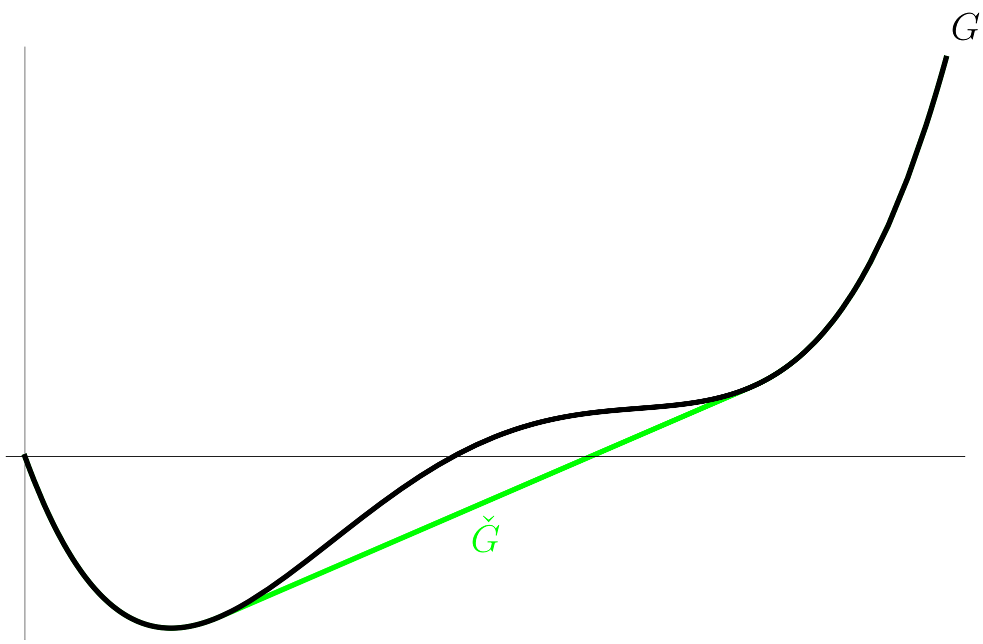

Figure 4. The function $G = \int g$ and its convex envelope $\check{G}$.

Unfortunately computing $\check{G}$ from G is not easy in general and gets worse in more than one dimension.

There is an important fact concerning the relation between G and $\check{G}$.

13 Fact *If x is an interior point of $[\underline{x}, \overline{x}]$, and if $G(x) > \check{G}(x)$, then $\check{G}$ is affine on a neighborhood of x, so $\check{G}''(x) = 0$.*

7.2 A change of variable

The ironing procedure described above balances the expected value only for a uniform distribution. So we start with a change of variable that will make the new types uniformly distributed.

Define the **fractile** φ of θ by

$$\varphi(\theta) = F(\theta).$$

Then φ is uniformly distributed on $[0,1]$,

$$\text{Prob}[\varphi \leqslant t] = \text{Prob}\big[F(\theta) \leqslant t\big] = \text{Prob}\big[\theta \leqslant F^{-1}(t)\big] = F\big(F^{-1}(t)\big) = t,$$

and

$$\theta = F^{-1}\big(\varphi(\theta)\big).$$

(Since F is strictly increasing, F^{-1} exists.[9])

7.3 The transformed problem

We now transform the problem to deal with fractiles. By transforming variables, and writing

$$\boldsymbol{\gamma}_i(\boldsymbol{\varphi}) = \boldsymbol{q}_i\big(F^{-1}(\boldsymbol{\varphi})\big) \quad \text{and} \quad \hat{\tau}(\varphi) = \tau\big(F^{-1}(\tau)\big)$$

the optimization problem becomes

$$\text{maximize} \quad \int_{[0,1]^N} \left\{ \sum_{i=1}^{N} \hat{\tau}(\boldsymbol{\varphi}_i)\boldsymbol{\gamma}_i(\boldsymbol{\varphi}) \right\} d\boldsymbol{\varphi}$$

subject to the following constraints: $\qquad (\boldsymbol{\Phi})$

1. $\boldsymbol{\gamma}_i(\boldsymbol{\varphi}) \geqslant 0$,
2. $\sum_{i=1}^{N} \boldsymbol{\gamma}_i(\boldsymbol{\varphi}) = 1$.
3. The reduced form $\boldsymbol{\Gamma}$ of $\boldsymbol{\gamma}$ is nondecreasing.

[9] Even if F is not continuous or strictly increasing, we can define

$$\varphi(\theta) = \sup\{z \in \boldsymbol{R} : F(z) \leqslant \theta\} := \widehat{F}(\theta)$$

and it still follows that $\text{Prob}\big[\varphi(\theta) \leqslant t\big] = t$. Then defining

$$F^{\leftarrow}(\varphi) = \inf\{x \in \boldsymbol{R} : F(x) \geqslant \varphi\}.$$

we have that $\widehat{F}(\theta)$ is uniformly distributed, $F^{\leftarrow}\big(\widehat{F}(\theta)\big) = \theta$, $\widehat{F}\big(F^{\leftarrow}(\varphi)\big) = \varphi$, and if φ is uniformly distributed $F^{\leftarrow}(\varphi)$ has cdf F. Moreover if F is continuous and strictly increasing, $\widehat{F} = F$ and $F^{\leftarrow} = F^{-1}$.

Note that since F and F^{-1} are strictly increasing, $\boldsymbol{\gamma}$ is nondecreasing if and only if $\boldsymbol{q}$ is. If τ is not strictly increasing, then neither is $\hat{\tau}$, but at least φ is uniformly distributed.

I need better notation!

Consider the ironed out version of the problem, where $\hat{\tau}$ is replaced by $\tilde{\tau}$ and the nondecreasing requirement is dropped. We know the solution to this problem is to assign the object to i with the highest $\tilde{\tau}(\boldsymbol{\varphi}_i)$ provided $\tilde{\tau}(\boldsymbol{\varphi}_i)$, and that doing so is incentive compatible since $\tilde{\tau}$ is nondecreasing.

$$\text{maximize} \quad \int_{[0,1]^N} \left\{ \sum_{i=1}^{N} \tilde{\tau}(\boldsymbol{\varphi}_i)\boldsymbol{\gamma}_i(\boldsymbol{\varphi}) \right\} d\boldsymbol{\varphi}$$

subject to the following constraints: $(\boldsymbol{I})$

1. $\boldsymbol{\gamma}_i(\boldsymbol{\varphi}) \geqslant 0$,
2. $\sum_{i=1}^{N} \boldsymbol{\gamma}_i(\boldsymbol{\varphi}) = 1$.

14 Lemma *The allocation rule that solves problem* $\boldsymbol{I}$ *is given by*

$$\boldsymbol{\gamma}_i^*(\boldsymbol{\varphi}) = \begin{cases} \dfrac{1}{|\tilde{M}(\boldsymbol{\varphi})|} & \tilde{\tau}^{\max}(\boldsymbol{\varphi}) > 0 \text{ and } i \in \tilde{M}(\boldsymbol{\varphi}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{M}(\boldsymbol{\varphi})$ *is the set of* i *for which* $\tilde{\tau}(\boldsymbol{\varphi}_i)$ *is a maximum and* $\tilde{\tau}^{\max}(\boldsymbol{\varphi})$ *is the maximum value.*

The next result is the key.

15 Lemma *The allocation rule that solves problem* $\boldsymbol{I}$ *is also solves problem* $\boldsymbol{\Phi}$.

Proof: Rewrite the integrand of problem ($\boldsymbol{\Phi}$) as

$$\begin{aligned} &\int_{[0,1]^N} \left\{ \sum_{i=1}^{N} \hat{\tau}(\varphi_i)\gamma_i(\varphi) \right\} d\varphi \\ &= \int_{[0,1]^N} \left\{ \sum_{i=1}^{N} \tilde{\tau}(\varphi_i)\gamma_i(\varphi) \right\} d\varphi - \int_{[0,1]^N} \left\{ \sum_{i=1}^{N} [\hat{\tau}(\varphi_i) - \tilde{\tau}(\varphi_i)]\gamma_i(\varphi) \right\} d\varphi \end{aligned}$$

and let's examine the second term.

The notation here is really bad.

$$\begin{aligned}&\int_{[0,1]^N} \left\{ \sum_{i=1}^N [\hat{\tau}(\varphi_i) - \tilde{\tau}(\varphi_i)] \gamma_i(\varphi) \right\} d\varphi \\ &\quad = \int_{[0,1]} \int_{[0,1]^{N-1}} \left\{ \sum_{i=1}^N [\hat{\tau}(\varphi_i) - \tilde{\tau}(\varphi_i)] \gamma_i(\varphi) \right\} d\varphi_{-i}\, d\varphi_i \\ &\quad = \int_{[0,1]} [\hat{\tau}(\varphi_i) - \tilde{\tau}(\varphi_i)] \Gamma(\varphi_i)\, d\varphi_i\end{aligned}$$

where Γ is the reduced form of γ. Integrating by parts yields

$$= \left(\int \hat{\tau} - \int \tilde{\tau} \right)\Big|_0^1 - \int_0^1 \left(\int \hat{\tau} - \int \tilde{\tau} \right)(\varphi)\, d\Gamma(\varphi).$$

Since $\tilde{\tau}$ is the ironed version of $\hat{\tau}$, we have that $\int \tilde{\tau}$ is the convex envelope of $\int \hat{\tau}$,

$$\left(\int \hat{\tau} - \int \tilde{\tau} \right)\Big|_0^1 = 0,$$

and

$$\int \hat{\tau}(\varphi) - \int \tilde{\tau}(\varphi) \geqslant 0,$$

so for any nondecreasing Γ, we also have

$$\int_0^1 \left(\int \hat{\tau} - \int \tilde{\tau} \right)(\varphi)\, d\Gamma(\varphi) \geqslant 0.$$

So

$$\begin{aligned}&\int_{[0,1]^N} \left\{ \sum_{i=1}^N \hat{\tau}(\varphi_i) \gamma_i(\varphi) \right\} d\varphi \\ &\quad = \underbrace{\int_{[0,1]^N} \left\{ \sum_{i=1}^N \tilde{\tau}(\varphi_i) \gamma_i(\varphi) \right\} d\varphi}_{A} - \underbrace{\int_0^1 \left(\int \hat{\tau} - \int \tilde{\tau} \right)(\varphi)\, d\Gamma(\varphi)}_{B}.\end{aligned} \tag{10}$$

For *any* nondecreasing Γ, term $(B) \geqslant 0$, and we now proceed to show that it equals zero for Γ^*. So recall from Fact 13 that if $\int \hat{\tau}(\varphi) > \int \tilde{\tau}(\varphi)$ (it's greater than its convex envelope), the ironed version $\tilde{\tau}$ is flat in a neighborhood of φ. But Γ^* is an increasing function of $\tilde{\tau}$, so it is flat too. That is,

$$\int \hat{\tau}(\varphi) > \int \tilde{\tau}(\varphi) > 0 \implies \Gamma^{*\prime}(\varphi) = 0.$$

Thus

$$\int_0^1 \left(\int \hat{\tau} - \int \tilde{\tau} \right)(\varphi)\, d\Gamma^*(\varphi) = \int_0^1 \left(\int \hat{\tau} - \int \tilde{\tau} \right)(\varphi) \Gamma^{*\prime}(\varphi)\, d\varphi = 0.$$

That is, γ^* maximizes term (A), and among nondecreasing reduced forms it minimizes term (B), so it is optimal for the original problem! ∎

An important feature of the solution to the general problem is that there is now a nontrivial probability of a tie. Where $\tilde{\tau}$ is flat, all the types are given equal priority.

7.4 Changing the variables back

By translating back to the oringinal types, we can easily characterize the revenue-maximizing auction.

Define

$$\overline{\tau}(\theta) = \tilde{\tau}\big(F^{-1}(\theta)\big).$$

16 Theorem *The revenue-maximizing auction* $\boldsymbol{q}$ *is given by*

$$\boldsymbol{q}_i^*(\boldsymbol{\theta}) = \begin{cases} \dfrac{1}{|\overline{M}(\boldsymbol{\theta})|} & \overline{\tau}^{\max}(\boldsymbol{\theta}) > 0 \text{ and } i \in \overline{M}(\boldsymbol{\theta}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{M}(\boldsymbol{\theta})$ *is the set of* i *for which* $\overline{\tau}(\boldsymbol{\theta}_i)$ *is a maximum and* $\overline{\tau}^{\max}(\boldsymbol{\theta})$ *is the maximum value.*

Myerson interprets $\overline{\tau}(\theta)$ as type θ's **priority level**. In the general case, the priority level is a complicated function of the value, being gotten by ironing the virtual type as a function of the fractile.

Unfortunately, while the reduced form of the transfer is still calculated according to (4), the price function is not very straightforward, due to the positive probability of ties. Suppose that bidder i wins the object with type θ_i and priority class $\overline{\tau}(\theta_i)$. In order for the auction to be incentive compatible, he must be willing to the price demanded. That is, the price can be no higher than the value of the lowest-value type with priority $\overline{\tau}(\theta_i)$. In fact, the price can be no higher than the value of the lowest-value type with priority $\overline{\tau}\big(\overline{\mu}(\theta_i)\big)$. That is, the price the winner pays is the lowest-value in the second-highest-priority class (which may with positive probability be the highest-priority class). Elaborate on this.

Bibliography

[1] K. C. Border. 1991. Implementation of reduced form auctions: A geometric approach. *Econometrica* 59:1175–1187.

[2] M. Harris and A. Raviv. 1981. Allocation mechanisms and the design of auctions. *Econometrica* 49:1477–1499.

[3] E. Maskin and J. Riley. 1984. Optimal auctions with risk averse buyers. *Econometrica* 52:1473–1518.

[4] S. A. Matthews. 1983. Selling to risk averse buyers with unobservable tastes. *Journal of Economic Theory* 30:370–400.

[5] ——— . 1984. On the implementability of reduced form auctions. *Econometrica* 52:1519–1522.

[6] J. Moore. 1984. Global incentive constraints in auction design. *Econometrica* 52:1523–1536.

[7] M. Mussa and S. Rosen. 1978. Monopoly and product quality. *Journal of Economic Theory* 18:301–317.

[8] R. B. Myerson. 1981. Optimal auction design. *Mathematics of Operations Research* 6:58–73.

[9] R. T. Rockafellar. 1970. *Convex analysis*. Princeton: Princeton University Press.