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Adverse selection in insurance markets

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This note is based on Michael Rothschild and Joseph Stiglitz [2], who argued that in the presence of *adverse selection*, markets for insurance were not guaranteed to deliver efficient outcomes, nor even to have equilibria. But first we review some background material.

1 Expected utility hypothesis

The standard model of choice over random variables is the expected utility (EU) model, which posits that a decision maker (dm) ranks random variables according to the expected value of their **Bernoulli utility** function u. That is, X is preferred to Y if $\boldsymbol{E}\, u(X) \geqslant \boldsymbol{E}\, u(Y)$.

Two Bernoulli utilities u and v represent the same preference ranking if and only if there are real numbers $a > 0$ and b satisfying $u(x) = av(x) + b$. That is, *Bernoulli utilities are unique up to positive affine transformation.*

See my on-line notes [1].

2 Risk aversion in the EU model

Risk aversion is the (weak) preference for $\boldsymbol{E}\, X$ for sure to X for all nondegenerate random variables X. That is,

$$u(\boldsymbol{E}\, X) \geqslant \boldsymbol{E}\, u(X).$$

In particular, if an EU dm with utility u is risk averse, and X assumes the values x and y with probabilities $1 - p$ and p respectively, then

$$u\Big((1-p)x + py\Big) \geqslant u\Big((1-p)x + py\Big).$$

In other words, u is **concave**. Conversely if u is concave, then the dm is risk averse, which is a mathematical result known as *Jensen's inequality*.

In practice, it is easiest to identify concave functions by their derivatives. A differentiable utility u is concave if and only $u'(x)$ is a monotone decreasing function of x. A twice-differentiable utility u is concave if and only $u''(x) \leqslant 0$ for all x. Note that linear functions are concave. A dm with a linear utility is **risk neutral** and ranks random variables according to their expectation.

3 State preference diagrams

A two-valued random variable can be represented as a point (x_a, x_b) in $\boldsymbol{R}^2$ (the value in event a is x_a and in event b is x_b). The diagonal $\{(x,x) : x \in \boldsymbol{R}\}$ is called the **certainty line**, the value of X is the same in either event.

An **indifference curve** is a set of random variables with the same expected utility. That is, the set of pairs (x, y) such that

$$p_a u(x) + p_b u(y) = \text{constant},$$

where p_a is the probability of event a, etc. For each x, let $\hat{y}(x)$ satisfy

$$p_a u(x) + p_b u\Big(\hat{y}(x)\Big) = \text{constant}.$$

Since the right-hand side is independent of x, its derivative with respect to x must be zero. That is,

$$p_a u'(x) + p_b u'\Big(\hat{y}(x)\Big)\hat{y}'(x) = 0,$$

so the slope $\hat{y}'$ of the indifference curve is

$$\hat{y}'(x) = -\frac{p_a u'(x)}{p_b u'\Big(\hat{y}(x)\Big)}.$$

Along the certainty line we have $\hat{y}(x) = x$, so the slope there is just $-p_a/p_b$.

3.1 Bets on a

A **bet on $\boldsymbol{a}$** is a random variable with $x_a > 0$ and $x_b < 0$. A bet is **fair** if its expectation is zero, which entails

$$p_a x_a + p_b x_b = 0, \qquad \text{or} \qquad -\frac{p_a}{p_b} = \frac{x_b}{x_a}.$$

A bet is **favorable** if

$$p_a x_a + p_b x_b > 0, \qquad \text{or} \qquad -\frac{p_a}{p_b} < \frac{x_b}{x_a}.$$

Suppose a risk averse EU dm with wealth w (the point $w = (w, w)$ on the certainty line) is offered the favorable bet x. See Figure 1. If his indifference curve is as drawn, he will not want to take the bet, since it would put him on a lower indifference curve. But since his indifference curve has slope $-p_a/p_b$ at w, the line segment joining w and $w + x$ crosses higher indifference curves so for small $\lambda > 0$, the point $w + \lambda x$ is preferred to w. So the dm would prefer to be able to take the bet λx.

1 Proposition *A risk averse EU dm with a smooth Bernoulli utility will prefer to take a small part of any favorable bet.*

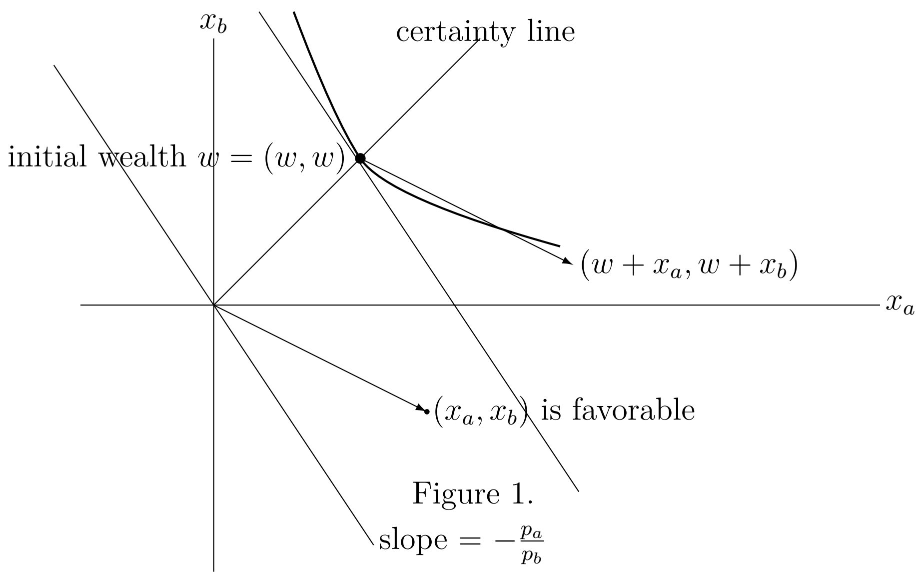

Figure 1.

4 Consumer types

We use a highly stylized model of insurance to starkly illustrate some of the key ideas. There are two **types** of insurance customers who are identical except for one trait—the probability that they will experience a loss. We assume that customers know their own type, but there is no way the insurance company can verify the type of a customer. This **asymmetric private information** is a source of problems in this market.

We consider only two states of the world, state 1 in which no loss occurs, so the wealth is w, and state 2, in which a loss of size c is suffered. Customers of type H are high-risk customers and have a probability p_H of a loss. Customers of type L are low-risk customers and have a probability p_L of a loss. Naturally,

$$1 > p_H > p_L > 0.$$

Assume the customers are EU decision makers with Bernoulli utility u. In the absence of insurance the expected utility of a type θ customer is

$$(1 - p_\theta)u(w) + p_\theta u(w - c),$$

where $\theta \in \Theta = \{L, H\}$.

A *state-preference diagram* is shown in Figure 2. Points in the plane represent random variables, that is, they represent the wealth in the two states of the world. The black dot is the *endowment point* $(w, w - c)$, so it lies below the *certainty line*. The red curve in the figure is an indifference curve of the High-risk type, and its slope at the certainty line is $-(1 - p_H)/p_H$.

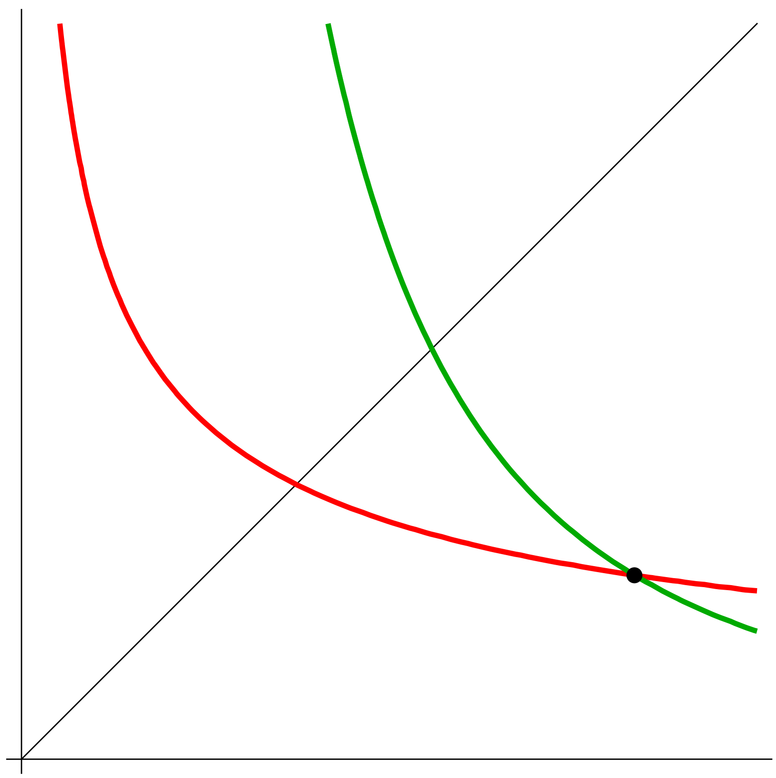

Figure 2. The black dot is the initial endowment absent insurance; the red indifference curve is for the High-risk type; the green indifference curve is for the Low-risk type.

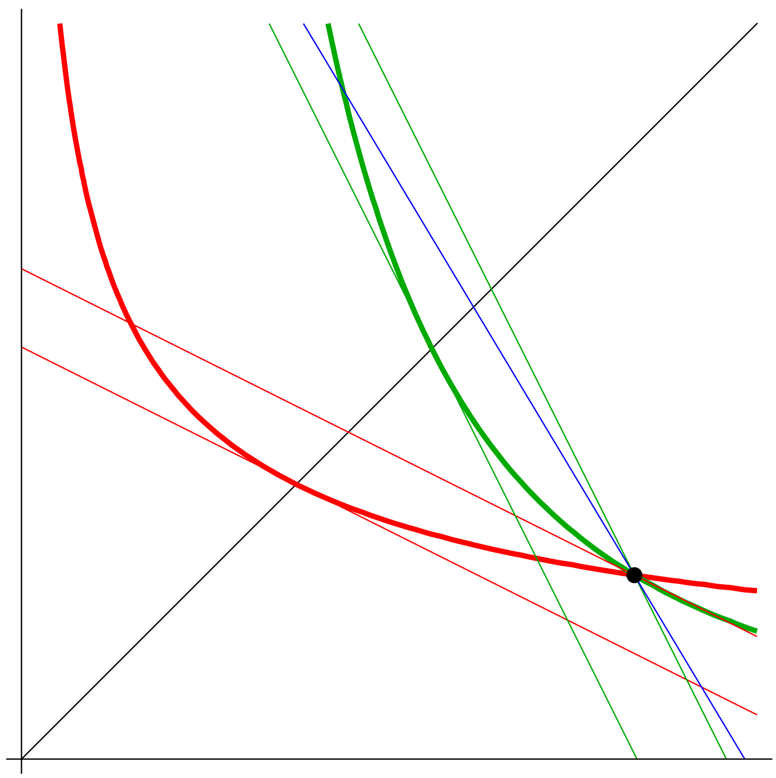

Figure 3. The black dot is the initial endowment absent insurance; the red lines are lines of equal expected value for p_H; the green lines are lines of equal expected value for p_L. The blue line is an iso-expected value line for p_A

5 Insurance policies

An **insurance policy** Q is characterized by two parameters, the premium π and the benefit b that is payed in case of a loss. Since in our simple model all consumers are identical in terms of their initial wealth and size of the loss, it is more convenient to represent a policy by its **result**,

$$X = (w - \pi, w - \pi + b - c).$$

The slope of the line segment connecting this point to the initial endowment is thus $-(b - \pi)/\pi$.

If p is the probability that a policyholder experiences a loss, the expected profit of a policy $Q = (\pi, b)$ to the insurance company is

$$\pi - pb.$$

The expected profit is nonnegative if and only if

$$\frac{1-p}{p} \geqslant \frac{b-\pi}{\pi}.$$

Thus a policy Q has a positive expected profit if and only if its result lies below the line through the endowment having slope $-(1-p)/p$), where p is the probability of a policyholder loss.

Figure 3 adds lines of equal expected value for the two types through the endowment. These lines indicate indifference curves for a risk-neutral insurance company.

Let λ denote the fraction of the population that is High-risk. The **average probability of a loss** is then

$$p_A = \lambda p_H + (1 - \lambda)p_L.$$

The iso-expected valued line for the average probability of loss is shown in Figure 4. Note that in this example the full-insurance policy for the average customer (**FIPAC**), whose result is represented by the blue dot, is preferred to the initial endowment by both types H and L.

6 Equilibrium concept

An **equilibrium** in this market consists of a partition $\mathfrak{T}$ of the type set Θ, and a list of pairs

$$(Q_T, T), \quad T \in \mathfrak{T},$$

where Q_T is the policy purchased by consumers with type $\theta \in T$, such that

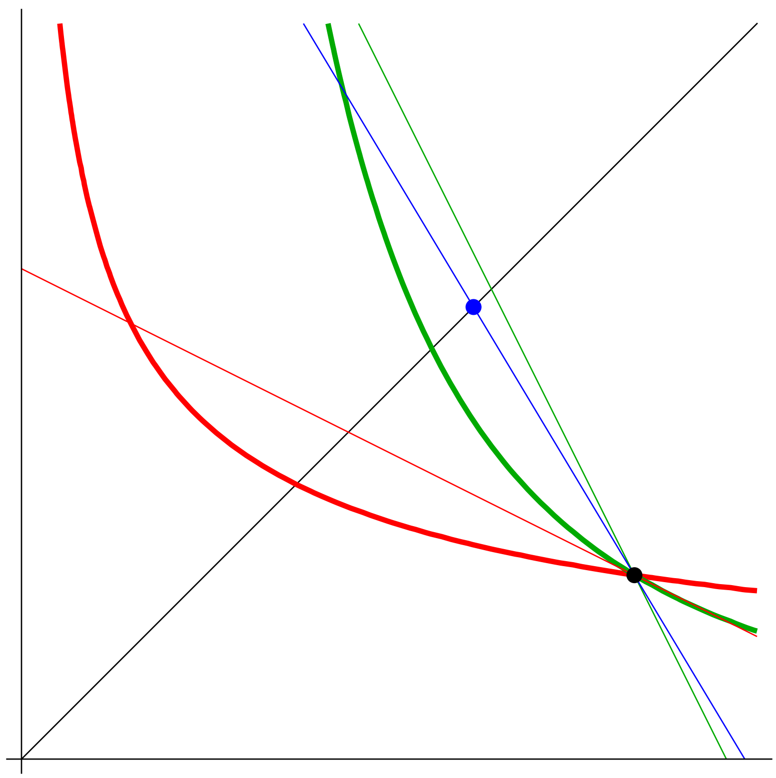

Figure 4. The black dot is the initial endowment absent insurance; the red line is an iso-expected value line for p_H; the green line is an iso-expected value line for p_L; and the blue line is an iso-expected value line for p_A.

Self-selection Each consumer with type θ in T prefers Q_T to any other policy. (Note that $Q = (0, 0)$, i.e., no insurance, is allowed to be one of the policies.)

Zero profit Each policy Q_T has expected profit zero, when the probability of a loss is the average probability of a loss for the set T.

Policy stability An insurer cannot make a positive expected profit by introducing a new policy. That is, there does not exist a policy Q' and set S of types such that S is the set of types θ who prefer Q' to Q_T, where $\theta \in T$, and Q' has positive expected profit when purchased by members of S.

In our simple model, there are two types of equilibria. A **separating equilibrium** has two policies Q_H and Q_L, where type H buys Q_H and type L buys Q_L. The policy Q_H has zero expected profit if the probability of loss is p_H and policy Q_L has zero expected profit if the probability of loss is p_L. The second kind of equilibrium is a **pooling equilibrium** with a single policy Q that is purchased by all consumers and has zero expected profit when the probability of loss is $p_A = \lambda p_H + (1 - \lambda)p_L$.

7 Non-existence of pooling equilibrium

Call the pooling policy **FIPAC** (for full insurance policy for average customer). One might be tempted to think that competition among risk-neutral insurers would lead to the pooling policy as the market equilibrium. After all, risk-averse customers prefer full insurance, and the insurance company breaks even in expected value. As Rothschild and Stiglitz pointed out, the problem with this is that it is possible to offer a new policy that will make money by siphoning off the Low-risk customers from the **FIPAC**. That is, there is a policy (many, in fact) that is preferred to **FIPAC** by the Low-risk types, but is not preferred by the High-risk types, and has positive expected value for the insurance company when purchased only by Low-risk types. The orange region in Figure 5 shows the set of results of such policies. This siphoning-off of the Low-risk types leaves, only the High-risk types purchasing **FIPAC**, which now has a negative expected value to the insurance company. This is known as **adverse selection** in the insurance industry.

8 Separating equilibrium

So what kind of policies can be supported? Figure 6 shows a separating market equilibrium in which the insurance industry offers two policies. The red dot is the result of full insurance to the High-risk types (**FIH**) and has expected value zero at p_H. The green dot is the result of partial insurance to the Low-risk types (**PIL**) and has expected value zero. In this example, the **PIL** result is preferred to any result on the blue line, which would pool High and Low risks into an average risk. The **PIL** is the best policy the market can deliver to the Low-risk types, so the policy offerings are stable.

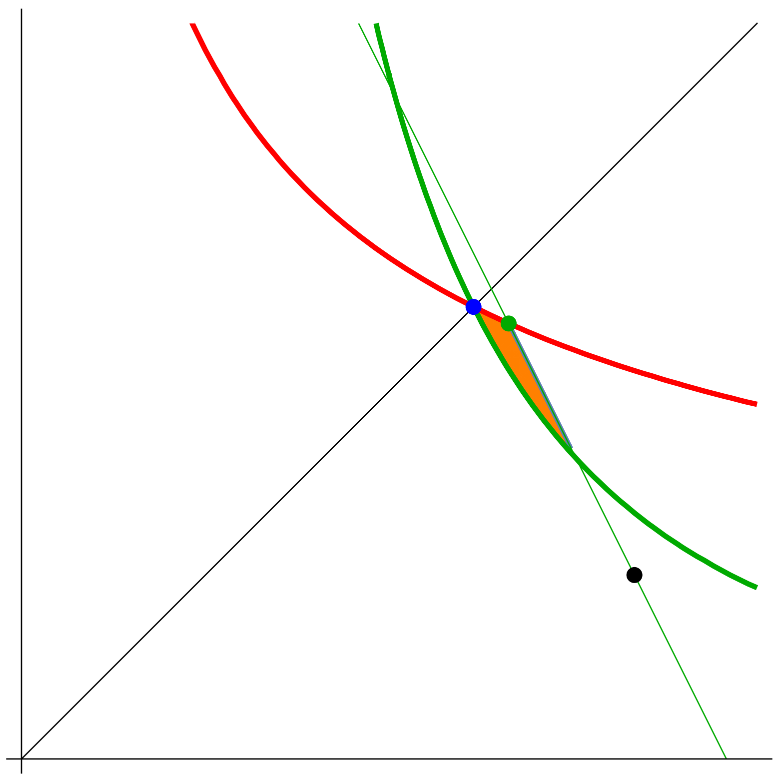

Figure 5. The orange region is preferred by type L to the result of **FIPAC**, the blue dot. It is not preferred by type H, and lies below the green line so it is profitable to sell to type L.

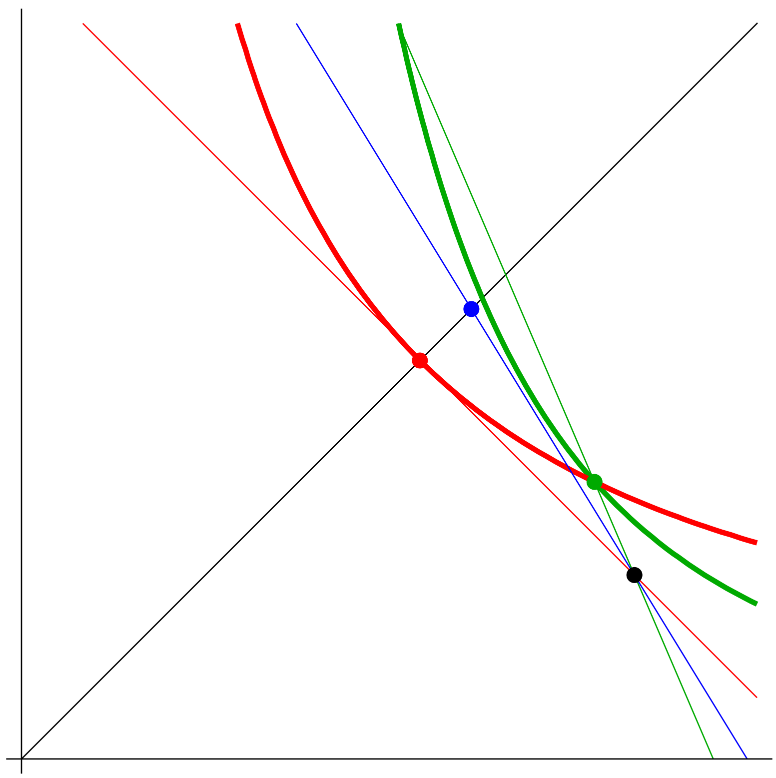

Figure 6. Separating Equilibrium

9 Non-existence of any equilibrium

Figure 7 shows a market in which the separating equilibrium described above does not exist. The red dot is again the result of full insurance to the High-risk types (FIH) and

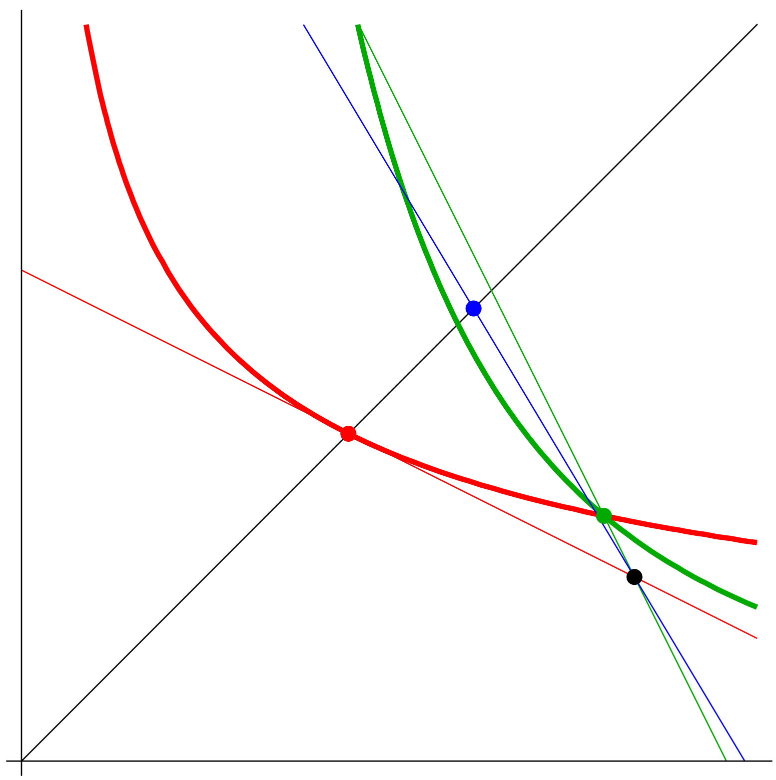

Figure 7. Failure of separating equilibrium.

has expected value zero at p_H. The green dot is the result of the most favorable partial insurance to the Low-risk types (**PIL**) and has expected value zero. In this example, the **PIL** result is inferior to the blue point, which would pool High and Low risks into an average risk. This means that the blue policy would be bought by everyone if it were offered, so the policy offerings are not stable—a minor perturbation of the **FIPAC** will earn strictly positive profits and siphon off both types.

A Parameters for the examples

The parameters for the examples were chosen to yield legible figures, not for "realism."

Example	Utility	p_H	p_L	λ	w	c
Section 8	$u(x) = \ln x$	1/2	3/10	2/5	10	7
Section 9	$u(x) = \ln x$	2/3	1/3	1/8	10	7

References

[1] K. C. Border. 1984–2013. Aspects of normative decision theory. http://www.its.caltech.edu/~kcborder/Notes/Decision.pdf

[2] M. Rothschild and J. Stiglitz. 1976. Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *Quarterly Journal of Economics* 90:629–649. http://www.jstor.org/stable/1885326