Designing Stable Mechanisms for Economic Environments

P.J. Healy (OSU)  L. Mathevet (UT Austin)

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SAET Ischia
Motivation

Behavioral  Mechanism Design
Objective: Design a game so that agents reach some desired objective in equilibrium
Behavioral Mechanism Design

Motivation

1. Starting point: Groves & Ledyard 1977
1a. Nash implementation
1b. ‘Economic’ Environments:
   Continuity, complexity (message space size), etc.
   Differentiability

2. Lesson from experiments: Stability matters
   • Chen & Plott 1996: ‘stability’ matters
   • Chen & Tang 1998: supermodularity
   • Arifovic & Ledyard 2003: something weaker
   • Healy 2006: dominant diagonal? specific dynamic?
   • Arifovic & Ledyard 2008: even weaker...
   • Current state of knowledge: supermodularity is sufficient.
This Paper

1. Understand how to develop G-L-like mechs.
2. Add ‘stability’ to the design constraints.

- Economic Environment: Two commodities
  \( x_i = \text{numéraire}, y_i = \text{private or public good} \)
- SCC: Walrasian or Lindahl equilibria (Hurwicz ’79)
- Continuously diff’bl mechanisms with ‘small’ strategy spaces

**Theorem 1:** Green-Laffont-type necessary cond’n:
\[
\text{tax}_i(m) = \text{price}_i(m_{-i})y_i(m)
\]

**Theorem 2:** Impossibility results for 1-dimensional \( m \):
- WE: No mechanism. LE: No ‘stable’ mechanism.

**Theorem 3:** Design stable mechanisms by adding a dimension to \( \mathcal{M} \)
The Economic Environment

- Agents: \( i \in \{1, 2, \ldots, n\} \).
- Work with net trades; no consumption set boundaries
- Agent \( i \)'s endowment: \( \omega_i = (0, 0) \).
- Net trade vector \( z_i = (x_i, y_i) \)
  - \( x_i \in \mathbb{R} \): numeraire good
  - \( y_i \in \mathbb{R} \): non-numeraire good (pub. or pvt)
- Agent \( i \)'s type: \( \theta_i \in \Theta_i \) (complete information.)
- Later: QSL Preferences: \( v_i(y_i|\theta_i) + x_i \).
  - \( v_i \) is differentiable, strictly concave.
A Walrasian equilibrium is \((z^*, p^*)\) such that

1. each \(z_i^*\) maximizes \(u_i\) s.t. \(x_i + p^* y_i \leq 0\), and
2. \(\sum_i z_i^* = 0\).

Public good: Set \(c(y) = \kappa y\).

A Lindahl equilibrium is \((z^*, p_1^*, \ldots, p_n^*)\) such that

1. each \(z_i^*\) maximizes \(u_i\) s.t. \(x_i + p_i^* y_i \leq 0\),
2. \((\sum_i p_i^*) y - \kappa y\) is maximized at \(y^*\), and
3. \(y_i^* = y^* \ \forall i\) and \(\sum_i x_i^* + \kappa y^* = \sum_i \omega_i\).

Walrasian and Lindahl correspondences: \(f : \Theta \rightarrow Z\)
Mechanisms

- Real-message mechanisms:
  - Strategy space: \( M_i = \mathbb{R}^{K_i} \forall i \)
  - Outcome functions: \((y_i(m), x_i(m))_i\)

- Given a mechanism \((\mathcal{M}, h)\), the Nash correspondence \(\nu : \Theta \rightarrow \mathcal{M}\) identifies the set of Nash equilibria for each \(\theta\).

- A mechanism \((\mathcal{M}, h)\) implements a social choice correspondence if \(h(\nu(\theta)) = f(\theta)\) for all \(\theta\).
Supermodularity & Stability

Previous literature: supermodularity $\Rightarrow$ stability.

Supermodularity:

1. $\frac{\partial^2 u_i}{\partial m_{ik} \partial m_{il}} \geq 0$ for all $i, k \neq l$.
2. $\frac{\partial^2 u_i}{\partial m_{ik} \partial m_{jl}} \geq 0$ for all $i \neq j, k, l$.
3. Strategy space is a closed interval.

Milgrom & Roberts: ‘adaptive dynamics’ converge to $[\text{NE}, \overline{\text{NE}}]

First 2 conditions: increasing BR curves.

Last condition: ignored in mechanism design!! Problem??
Both games are “supermodular”. Left game is stable, right is not. Slope of BR curves matters!
Unstable game: boundaries create ‘bad’ (stable) corner equilibria. ‘Stability’ property of supermodularity vacuous here.
“Counter-Example” Mechanism

Assume $v_i''(\cdot | \theta_i) \in [-M, 0)$ for all $\theta \in \Theta$. Choose

$$y(m) = \sum_{i=1}^{n/2} m_i - \sum_{n/2+1}^{n} m_i$$

$$q_i(m) = \begin{cases} 
\frac{\kappa}{n} - \gamma \sum_{j \neq \{i, i+\frac{n}{2}\}} m_j & \text{if } i \leq n/2 \\
\frac{\kappa}{n} + \gamma \sum_{j \neq \{i, i+\frac{n}{2}\}} m_j & \text{if } i > n/2.
\end{cases}$$

Supermodular if $\gamma > M$. But best response dynamic:

![Graph showing best response dynamics]
Van Essen’s suggestion:

- Can we make mechanisms with BR curves that are contraction mappings?
- \( \|BR(x) - BR(y)\| \leq \alpha \|x - y\| \) for \( \alpha \in (0, 1) \).
- For now, assume BR is single-valued.

**Definition**

A mechanism is **contractive** on \( \Theta \) if \( BR \) is single-valued and for every \( \theta \in \Theta \) there exists some \( \alpha \in (0, 1) \) such that for every \( m, m' \in M \),

\[
\|BR(m') - BR(m)\| \leq \alpha \|m' - m\|.
\]
Does Contractive Imply Stable?

Adaptive Best-Response (ABR) Dynamics:

**Theorem:** If \( \{m(t)\} \) is an ABR Dynamic and \( BR(\cdot) \) is contractive then \( m(t) \) converges to \( m^* \).
OK... how can we make a mechanism contractive? 
Step 1: Understand how mechanisms look & feel.

Trivial Observation:
Every mechanism’s numeraire outcome functions can be written as

\[ x_i(m) = -q_i(m_i) y_i(m) - g_i(m) \]

‘Price’ ‘Qty’ ‘Penalty’

Note: ‘Price-taking’ assumption
Some Existing P.G. Mechanisms

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>$y(m)$</th>
<th></th>
<th>Price: $q_i(m_{-i})$</th>
<th>Penalty: $g_i(m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Groves-Ledyard '77</td>
<td>$\sum_i m_i$</td>
<td>$\kappa/n$</td>
<td>$\frac{\gamma}{2} \left[ \frac{n-1}{n} (m_i - \bar{m}<em>{-i})^2 - \sigma(m</em>{-i}) \right]$</td>
<td></td>
</tr>
<tr>
<td>Walker '81</td>
<td>$\sum_i m_i$</td>
<td>$\kappa/n - m_{i-1} + m_{i+1}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Hurwicz '79</td>
<td>$r_i - \bar{r}_{-i}$</td>
<td>$\bar{s}_{-i}$</td>
<td>$(s_i - \bar{s}<em>{-i})^2 + H_i(m</em>{-i})$</td>
<td></td>
</tr>
<tr>
<td>Chen '03</td>
<td>$\sum_i r_i$</td>
<td>$\frac{\kappa}{n} - \gamma \sum_{j \neq i} r_j + \frac{\gamma}{n} \sum_{j \neq i} s_j$</td>
<td>$- \frac{1}{2} (s_i - y(m))^2 + \frac{\delta}{2} \sum_{j \neq i} (s_j - y(m))^2$</td>
<td></td>
</tr>
</tbody>
</table>

In all of these...
(1) agents are ‘price-taking’, and
(2) if it implements Lindahl, $g_i = 0$ in equilibrium.
Eerie Similarities

Why are these mechanisms so similar?

How do they work?

How much freedom is there to play with them?
The Graphical View

\[ z = \chi_{ik}(m_{ik} | m_{\cdot ik}) \]

\[ \chi_{ik}(m_{ik} ' | m_{\cdot ik}) = z' \]

What you can achieve by changing \( m_i \) (given \( m_{\cdot i} \))
The Local Price

\[ P_{ik}(m_{ik}'' | m_{-ik}) := -\frac{\partial x_i / \partial m_{ik}}{\partial y_i / \partial m_{ik}} \]

Slope of \( \chi_i \) is the ‘local price’.
Possible Nash equilibrium points given $u_i$ or $u'_i$. 
Possible Walrasian allocations given $u_i$ or $u_i'$. 
Nash Implementation

Triple tangency is necessary for NE outcome to be WE.
‘Bad’ Nash Equilibria

Rich-enough type space $\Rightarrow$ ANY $m$ is a NE.
‘Bad’ Nash Equilibria

But now the mechanism doesn’t implement Walrasian allocations!
The Necessary Condition

Only way to avoid ‘bad’ equilibria: $t_i(m) = q_i(m_{-i})y_i(m)$. 

\[ \chi_{ik}(m_{ik}|m_{-ik}) = p \]
Assumptions

Ready to formalize this theorem...

- A1: (Differentiability) $y_i(m), x_i(m)$ are all twice continuously differentiable.
- A2: (Responsive $y_i$) $\frac{\partial y_i(m)}{\partial m_{ik}}$ is bounded away from zero. (Keeps $\chi_{ik}$ from going vertical.)
- A3: (Rich Domain & Regularity) All $m$ are NE for some $\theta$. 
The Necessary Condition

Theorem

Take any type space $\Theta$ and 1-dimensional mechanism satisfying A1-A3. If the mechanism Nash implements the Walrasian or Lindahl allocations, it must be that

$$x_i(m) \equiv -q_i(m-i)y(m).$$

(Thus, $g_i(m) \equiv 0$.)

Intuition: $q_i$ is a ‘fixed’ price for $i$. Since $y_i$ is bijective in $m_i$, $i$ can pick any $y_i$. Thus, he picks

$$\max_{y_i} u_i(-q_i(m-i)y_i, y_i)$$
One-Dimensional Walrasian Mechanisms

Theorem

Under A1-A3 there do not exist any one-dimensional mechanisms that implement the Walrasian correspondence.

Proof.

• Need \( q_1(m_{-1}) \equiv q_2(m_{-2}) \equiv \ldots \equiv q_n(m_{-n}) \)
• Only possible if all \( q_i \) are constant.
• \( p(\Theta) \) is not a singleton; a contradiction.

cf. Reichelstein & Reiter & dimensionality results.
One-Dimensional Lindahl Mechanisms

Assumption (A4)
For all $\theta \in \Theta$, $u_i(x_i, y_i|\theta_i) = v_i(y_i|\theta_i) + x_i$
with $v_i' > 0$ and $v_i'' \in (-B, 1/B)$ for some $B > 0$.

Proposition
Under A1-A4 there are no one-dimensional contractive mechanisms that implement the Lindahl correspondence.
Necessary Conditions: More Dimensions

- Let $M_i = R_i \times S_i$ so that $y: R \rightarrow \mathbb{R}$.
- What $(r, s)$ can never be a Nash equilibrium?
- $U_i(r, s) = v_i(y(r)|\theta_i) - q_i(r, s)y(r) - g_i(r, s)$
- Thus, $s^*_i(r, s_{-i})$ solves $\min_{s_i} q_i(s, r) * y(r) + g_i(s, r)$.
- Designer can calculate NE of the ‘tax-minimizing game’ $\forall r$.

Note: $(r, s)$ is NOT a NE if:

1. $s$ is not a NE of the tax-minimizing game, or
2. $P_{ik}(r, s) \neq P_{il}(r, s)$ for some $i, k, l$.

Assumption (A3’)

*If $m$ does not satisfy either of the above then $m$ is a NE for some $\theta$.\*
More Dimensions

Theorem

Under A1, A2, and A3’, for any ‘regular’ NE \((r, s)\),

\[ x_i(r, s) = -q_i(r, s)y_i(r) - g_i(r, s), \]

where

\[ \frac{dq_i(r, s^*_i(r, s), s_{-i})}{dr_i} = 0 \]

and

\[ g_i(r, s) \equiv 0 \]

along the equilibrium manifold.
Stable Mechanism Recipe

Recipe for designing a contractive mechanism:

1. Need bounded concavity ($v_i'' \in (-B, -1/B)$),
2. Start with $U_i(r, s) := v_i(y(r)) - q_i(r, s)y(r) - g_i(r, s)$,
3. Define best response functions $(\rho_i(r_{-i}, s_{-i}), \sigma_i(r_{-i}, s_{-i}))$.
4. Write down two FOCs:

   $$\frac{\partial U_i(\rho_i, \sigma_i, r_{-i}, s_{-i})}{\partial r_i} \equiv \frac{\partial U_i(\rho_i, \sigma_i, r_{-i}, s_{-i})}{\partial s_i} \equiv 0$$

5. Differentiate both sides (I.F.T.) and solve system for

   $$\left(\frac{\partial \rho_i}{\partial r_j}, \frac{\partial \rho_i}{\partial s_j}, \frac{\partial \sigma_i}{\partial r_j}, \frac{\partial \sigma_i}{\partial s_j}\right)$$
Stable Mechanism Recipe

For example:

\[
\frac{\partial \rho_i}{\partial r_j} = \frac{\partial^2 g_i}{\partial s_i^2} \left( -v_i'' \frac{\partial y}{\partial r_i} \frac{\partial y}{\partial r_j} + \frac{\partial y}{\partial r_i} \frac{\partial q_i}{\partial r_j} + \frac{\partial^2 g_i}{\partial r_i \partial r_j} \right) - \frac{\partial^2 g_i}{\partial r_i \partial s_i} \frac{\partial^2 g_i}{\partial s_i \partial r_j} \\
\left( \frac{\partial^2 g_i}{\partial r_i \partial s_i} \right)^2 + v_i'' \left( \frac{\partial y}{\partial r_i} \right)^2 \frac{\partial^2 g_i}{\partial s_i^2} - \frac{\partial^2 g_i}{\partial r_i^2} \frac{\partial^2 g_i}{\partial s_i^2}
\]

6 Find parameterized functions such that when some parameter gets big,
   a \( \sum_{j \neq i} \left( \left| \frac{\partial \rho_j}{\partial r_i} \right| + \left| \frac{\partial \sigma_j}{\partial r_i} \right| \right) < 1 \) and \( \sum_{j \neq i} \left( \left| \frac{\partial \rho_j}{\partial s_i} \right| + \left| \frac{\partial \sigma_j}{\partial s_i} \right| \right) < 1 \),
   b \( g_i = 0 \) in equilibrium, and
   c \( \sum_i q_i = \kappa \) in equilibrium.

7 Give up and hire an RA to do it.
A Contractive Lindahl Mechanism

\[ y(r) = \sum_i r_i \]

\[ q_i(r_{-i}, s_i) = \frac{\kappa}{n} + \frac{1}{\delta} (r_{i-1} - r_{i+1}) + \delta \frac{n-1}{n^2} (s_{i-1} - \frac{1}{n} r_{i+1}) \]

\[ g_i(r, s) = \frac{1}{2} (s_i - \frac{1}{n} r_{i+1})^2 + \frac{\delta}{2} (s_{i-1} - \frac{1}{n} r_i)^2 \]

**Theorem**

This implements Lindahl equilibria. If \( \delta \) is sufficiently large it becomes contractive.

(In fact, this is a ‘stabilized’ Walker mechanism.)
A Contractive Walrasian Mechanism

To be announced.
A Contractive $\varepsilon$-Walrasian Mechanism

$$y_i(r) = (r_{i-1} - r_{i+1}) - \frac{\delta}{n} (s_{i+1} - \frac{n+1}{n} r_i)$$

$$q_i(s_{-i}) = \frac{1}{n-1} \sum_{j \neq i} s_j$$

$$g_i(r, s) = (s_i - \delta \frac{n+1}{n^2} \sum_j r_j)^2$$

Theorem

*For large $\delta$ this mechanism is contractive and implements allocations arbitrarily close to the Walrasian allocations.*
Notes on this Procedure

- Stability demands large parameter values. Is this useful?
- Can we make an anonymous contractive mechanism?
- Contractive $\Rightarrow$ unique equilibrium.
  - What if SCC isn’t single-valued?
  - Note: contractiveness depends on $\Theta$.
- Van Essen et al. experiments on “supermodularity”
- Fact remains: supermodularity $\Rightarrow$ stability in the lab
  - Why??
  - Were those mechs. contractive for the chosen prefs?
  - Is there something else about supermodularity?
Further reading:

- Reichelstein & Reiter 1988: Some of the same ideas.
- Brock 1980 & G-L 1987: Sufficiency
- Mathevet 2008: Supermodular Mechanism Design
- Van Essen 2009 & Van Essen, Lazzati & Walker 2009

- Ultimate goal: practical mechanism design
- Conversation between experiments & theory.
The End