ExpEcon Methods:
A Theory of Testing Theories

ECON 8877

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Introduction

A principal wants to learn *something* about the preferences of an agent, but *not* the whole ordering
(Why not? complexity, costs, privacy, etc.)

Example: NYC school match: only list favorite 12 schools

Which properties of preferences can be elicited in an incentive compatible way?

Leading Example:

$X = \{x, y, z\}$. Let xyz denote $x \succ y \succ z$, *e.g.* Assume strict prefs.

All orderings:

$$\{xyz, xzy, zxy, zyx, yzx, yxz\}$$

A simple elicitation mechanism:
Pick from $\{x, y\}$
Paid what you choose

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Pick from $\{x, y\}$
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This type space is *elicitable*. Truth FOSD's lie.

$$\{xyz, xzy, zxy, zyx, yzx, yxz\}$$

Mechanism:
Pick from $\{x, y\}$ *and* from $\{x, z\}$
We randomly pick *one* of your answers and pay it to you

$$\{\underbrace{\{xyz, xzy\}}_{\text{pick } x, x}, \underbrace{\{zxy\}}_{\text{pick } x, z}, \underbrace{\{zyx, yzx\}}_{\text{pick } y, z}, \underbrace{\{yxz\}}_{\text{pick } y, x}\}$$

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$$\{\underbrace{\{xyz, yxz\}}_{\text{dislike } z}, \underbrace{\{xzy, zxy\}}_{\text{dislike } y}, \underbrace{\{yzx, zyx\}}_{\text{dislike } x}\}$$

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There are **no** menus that generate this type space.

Generated by top *two* elements of X

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Mechanism:
Announce least favorite,
get paid 50-50 lottery over the other two options.

Results

Preview of Main Results:

Generated by top k elements $\Rightarrow$ elicitable $\Rightarrow$ "convex"

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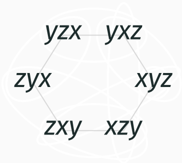

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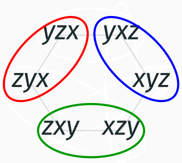

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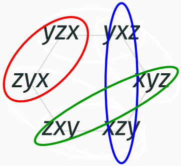

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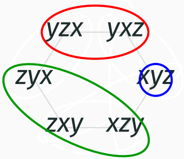

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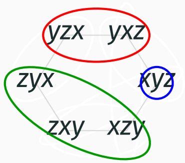

We get complete characterization when:

1. Restrict to neutral type spaces, or
2. Pay in acts, not lotteries (no objective probabilities)

Some related literature

- Osband (1985), Lambert-Pennock-Shoham (2008), Lambert (2018): Scoring rules to elicit properties of beliefs
- Gibbard (1977), Bahel and Sprumont (2019): Characterizing strategy-proof random mechanisms
- Carroll (2012), Saito (2013): Sufficiency of local IC constraints
- ACH (2018, 2020): Characterizing IC mechanisms in experimental environments

The General Model

Framework

- X - a finite set of alternatives
  - Typical elements: $x, y, z, w, \ldots$
- O - the set of strict orders over X
  - Typical elements: $\succeq, \succeq', \ldots$

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Definition
A *type space* $T = \{t_1, \ldots, t_k\}$ is a partition of O.

- A *type* is any $t \in T$, so $t = \{\succeq, \succeq', ..., \succeq''\}$

- Example: $t = \{\text{all} \succeq \text{satisfying the Independence axiom}\}$

- Notation: $t(\succeq) \in T$ is the type containing $\succeq$

Examples

$X = \{x, y, z\}$

- Entire ranking:
  $T = \{\{xyz\}, \{yxz\}, \{yzx\}, \{zyx\}, \{zxy\}, \{xzy\}\}$
- First-best:
  $T = \{\{xyz, xzy\}, \{yxz, yzx\}, \{zyx, zxy\}\}$
- Top-2:
  $T = \{\{xyz, yxz\}, \{xzy, zxy\}, \{zyx, yzx\}\}$
- Best from $\{x, y\}$:
  $T = \{\{xyz, xzy, zxy\}, \{zyx, yzx, yxz\}\}$
- Where you rank x:
  $T = \{\{xyz, xzy\}, \{zxy, yxz\}, \{yzx, zyx\}\}$
  (This type space is not "neutral". Labels matter.)

Mechanisms

$\Delta(X)$ is the set of lotteries on X

Definition
A T-mechanism is any $g : T \to \Delta(X)$.

- Why random payments?
  - Allows use of the RPS mechanism (and more)
  - With deterministic mechanisms very little can be elicited

Elicitable type spaces

Recall that p strictly FOSD q relative to $\succeq$ (written $p \succ^* q$) if

$$\forall x \in X \quad p(\{y : y \succeq x\}) \geq q(\{y : y \succeq x\})$$

with strict inequality for at least one x

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g is *IC* if for every $\succeq \in O$ and every $t \neq t(\succeq)$

$$g(t(\succeq)) \succ^* g(t).$$

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Definition
A type space T is *elicitable* if there exists an IC T-mechanism.

Goal: Characterize elicitable type spaces (spoiler: we can't)

Top elements of menus

"What's your favorite thing from X'?"

- Every menu $X' \subseteq X$ corresponds to a type space:

$$\succeq, \succeq' \in t \iff \succeq, \succeq' \text{ have the same favorite item in } X'$$

Examples:

$X' = \{x, y\} \implies T = \{\{xyz, xzy, zxy\}, \{zyx, yzx, yxz\}\}$

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- The (deterministic) mechanism that pays the revealed top element in X' is IC

RPS mechanisms

- One can elicit top elements of several menus $X_1, \ldots, X_l \subseteq X$

Examples:

$$X_1 = \{x, y, z\},\ X_2 = \{x, y\}$$
$$\implies \quad T = \{\{xyz, xzy\}, \{yzx, yxz\}, \{zxy\}, \{zyx\}\}$$

$$X_1 = \{x, y\},\ X_2 = \{x, z\},\ X_3 = \{y, z\}$$
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- The corresponding IC mechanism randomly chooses a menu and pays the announced top element
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What else is elicitable?

Top sets of menus

The top-2 type space $T = \{\{xyz, yxz\}, \{xzy, zxy\}, \{zyx, yzx\}\}$ does not reveal top elements of menus but is elicitable

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- How? If they announce "*x* and *y*" pay *x* and *y* with equal probability, and *z* with less probability.
- Every $X' \subseteq X$ and k defines a type space by

$$\succeq, \succeq' \in t \iff \succeq, \succeq' \text{ have the same top } k \text{ elements of } X'$$

- This is elicitable by paying the uniform lottery over the set of announced top-k elements
- Can elicit the top-k_i elements of $X_i \subseteq X$, $i = 1, \ldots, l$

Top sets of menus

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Anything else??

Example (based on Shapley, 1971)

$X = \{x, y, z, w\}$

Type space:
$\{xyzw, yxzw, xywz, yxwz\}$
$\{xzyw\}$, $\{xwyz\}$, $\{xzwy, xwzy\}$
$\{ywxz\}$, $\{yzxw\}$, $\{yzwx, ywzx\}$
$\{zxyw, zyxw\}$, $\{zywx, zwyx\}$, $\{zxwy, zwxy\}$
$\{wxyz, wyxz\}$, $\{wyzx, wzyx\}$, $\{wxzy, wzxy\}$

Claim
$\exists$ IC mechanism, but type space is not generated by top sets.

There is a close connection between IC mechanisms and convex TU cooperative games...

So far...

$$\{\text{all } T\}$$

$$\rotatebox{90}{$\supseteq$}$$

$$\color{red}{\{T : \text{elicitable}\}}$$

$$\rotatebox{90}{$\supsetneq$}$$

$$\{T : \text{generated by top sets}\}$$

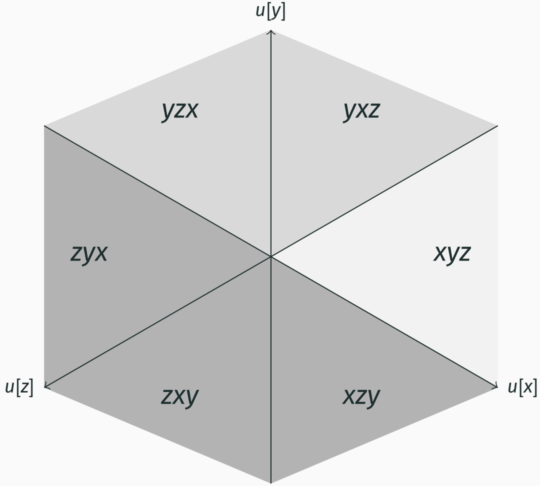
$$\rotatebox{90}{$\supsetneq$}$$

$$\{T : \text{generated by top elements}\}$$

A convex type space – example

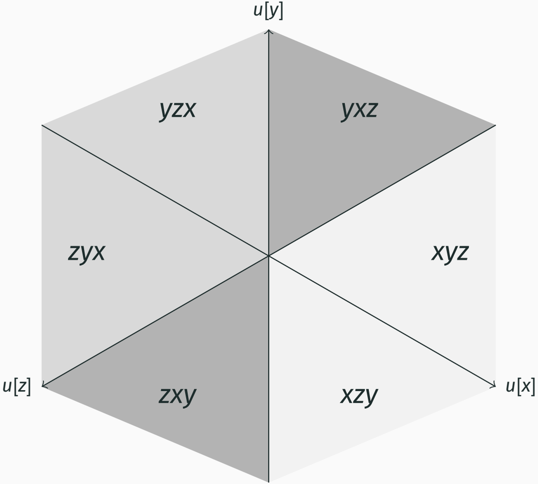
Necessary condition: **convex** type space

Example: $T = \{\{xyz\}, \{yxz, yzx\}, \{zyx, zxy, xzy\}\}$



A non-convex type space - example

Example of a non-convex type space:
$T = \{\{xyz, xzy\}, \{yxz, zxy\}, \{zyx, yzx\}\}$



Convexity is necessary

Proposition
If T is elicitable then it is convex.

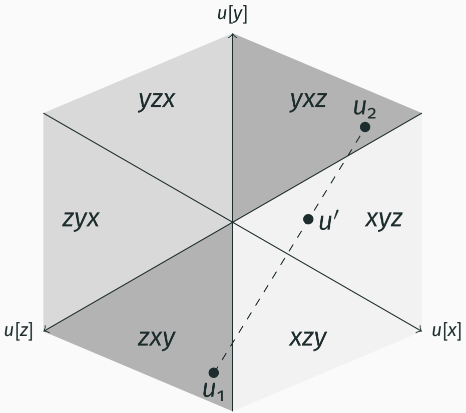
Ex: Where do you rank x?

$t = x$ is 2nd (dark gray)
$t' = x$ is 1st (off-white)

IC requires:

$\mathbb{E}_{g(t)}(u_1) > \mathbb{E}_{g(t')}(u_1)$
$\mathbb{E}_{g(t)}(u_2) > \mathbb{E}_{g(t')}(u_2)$

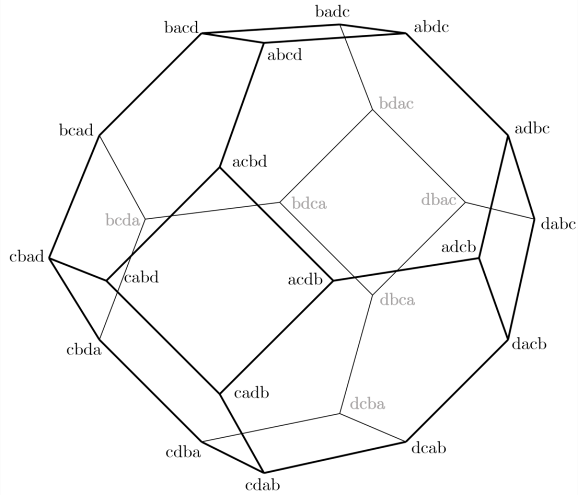
$\implies \mathbb{E}_{g(t)}(u') > \mathbb{E}_{g(t')}(u')$



Some non-convex type spaces

- Where do you rank x? (with $|X| \geq 3$)
- What is the kth ranked alternative for $1 < k < |X|$ (e.g. median)
- Any binary $T = \{t_1, t_2\}$, *except* $T = \{\{x \succeq y\},\ \{y \succeq x\}\}$.
  In particular, tests of most axioms of preferences!
  Usually: "If $x \succeq y$ then $w \succeq z$ (and $y \succeq x \Rightarrow z \succeq w$)"

Visualizing Convexity: The Permutohedron



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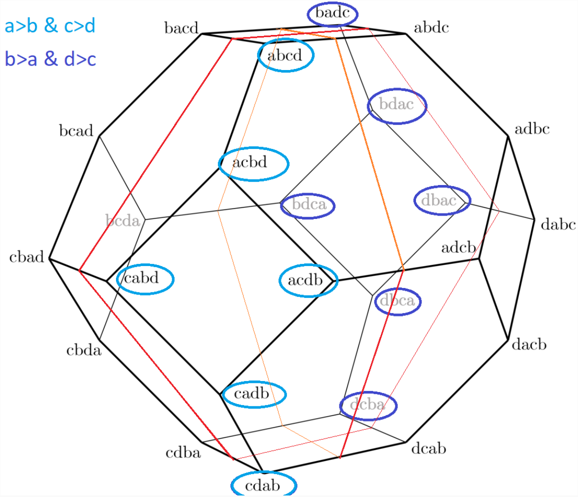


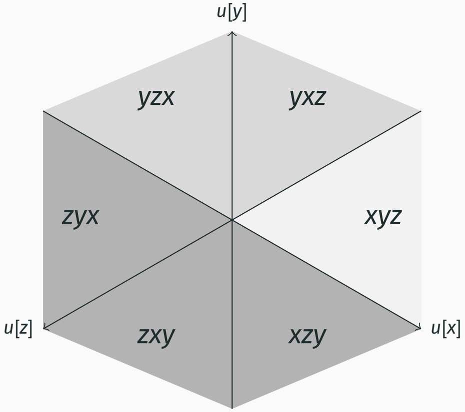
Convexity is not sufficient

$T = \{t_1 = \{xyz\}, t_2 = \{yxz, yzx\}, t_3 = \{zyx, zxy, xzy\}\}$

IC requires:

$g(t_1)(x) > g(t_2)(x)$
$g(t_2)(x) = g(t_3)(x)$
$g(t_3)(x) = g(t_1)(x)$

$\implies g(t_1)(x) > g(t_1)(x)$



Summary

$$\{\text{all } T\}$$
$$\supsetneq$$
$$\{T : \text{convex}\}$$
$$\supsetneq$$
$$\{T : \text{no bad cycles}\}$$
$$\supseteq$$
$$\{T : \text{elicitable}\}$$
$$\supsetneq$$
$$\{T : \text{generated by top sets}\}$$
$$\supsetneq$$
$$\{T : \text{generated by top elements}\}$$

Neutral type spaces

- Permutation: $\pi : X \to X$
- Let πT be T, but with every $\succeq$ permuted by π

Definition
T is *neutral* if $\pi T = T$ for every π.

Neutral: “What do you rank 3rd?”
Not: “Where do you rank x?”

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T is *neutral* if $\pi T = T$ for every π.

Neutral: "What do you rank 3rd?"
Not: "Where do you rank x?"

Proposition
Suppose T is neutral. Then the following are equivalent:
(1) T is elicitable
(2) T is convex
(3) T is generated by top sets

Neutral type spaces

$$\{\text{all } T\}$$
$$\cup\!\!\not\parallel$$
$$\{T : \text{convex}\}$$
$$\parallel$$
$$\{T : \text{elicitable}\}$$
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Robust elicitation

What if the agent has subjective beliefs about the likelihood of realizations of the randomization device (or other kinds of preferences over acts)?

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- Now payments are Savage acts, not lotteries.
- Can’t pay a uniform lottery b/c can’t control beliefs
- This kills our ability to elicit top sets

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Proposition
T is elicitable with acts iff it is generated by top elements.

Elicitation under acts

$$\{\text{all } T\}$$
$$\cup\!\!\!\!\not\,\,\,|$$
$$\{T : \text{convex}\}$$
$$\cup\!\!\!\!\not\,\,\,|$$
$$\{T : \text{elicitable with lotteries}\}$$
$$\cup\!\!\!\!\not\,\,\,|$$
$$\{T : \text{generated by top sets}\}$$
$$\cup\!\!\!\!\not\,\,\,|$$
$$\{T : \text{generated by top elements}\}$$
$$\parallel$$
$$\{T : \text{elicitable with acts}\}$$

Multiple agents

- $N = \{1, \ldots, n\}$ - agents
- T_i - agent's i type space
- $T = (T_1, \ldots, T_n)$ - a profile of type spaces
- $g : T \to \Delta(X)$ - a mechanism

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- T_i - agent's *i* type space
- $T = (T_1, \ldots, T_n)$ - a profile of type spaces
- $g : T \to \Delta(X)$ - a mechanism

Proposition
$T = (T_1, \ldots, T_n)$ is dominant-strategy-elicitable iff each T_i is elicitable.

Conclusion

- We formulate a notion of elicitability for properties of preferences
- Some necessary conditions and some sufficient conditions for elicitability, but no characterization
- We do have a characterization for neutral type spaces and for robust elicitation (acts)
- Potential extensions: Weak orders, infinite sets of alternatives, domain restrictions,...

Thank You!