ExpEcon Methods: A Decision Theory Primer

ECON 8877
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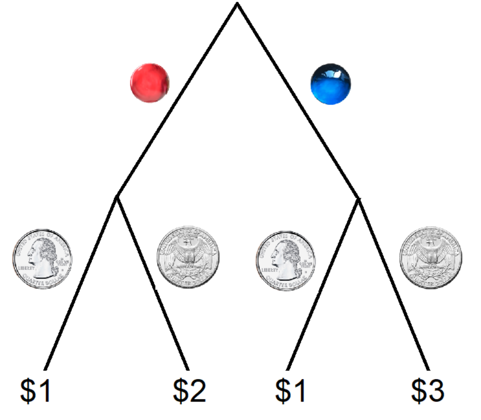
Differing Frameworks

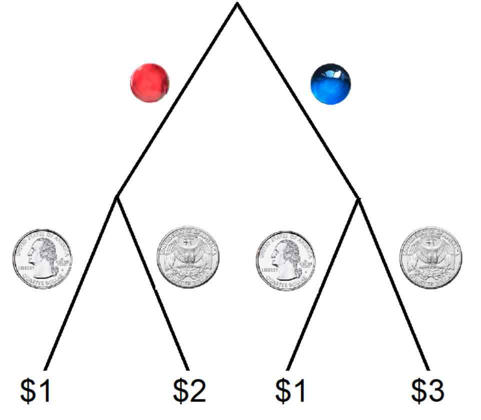
- The theory of incentives is built on classic decision theory frameworks
- Each models uncertainty in a different way
- Difference: which probabilities are “common knowledge”?
- Each has different axioms
    - So, sufficient conditions for an IC experiment will differ
- Historical confusion among experimentalists due to using different/unclear frameworks

An Example



How to model this gamble?

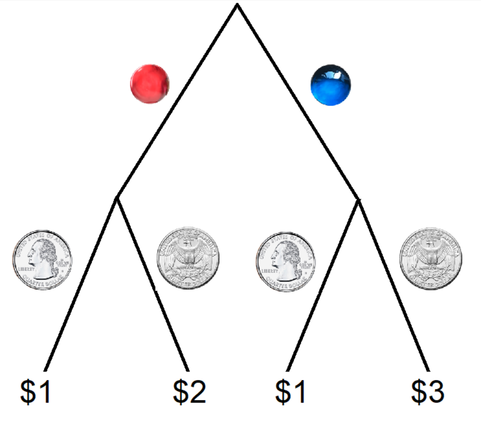
An Example



von Neumann-Morgenstern (vNM): All probabilities are known

Objective Lottery: $L = (\$1, 0.5; \$2, 0.25; \$3, 0.25)$

An Example

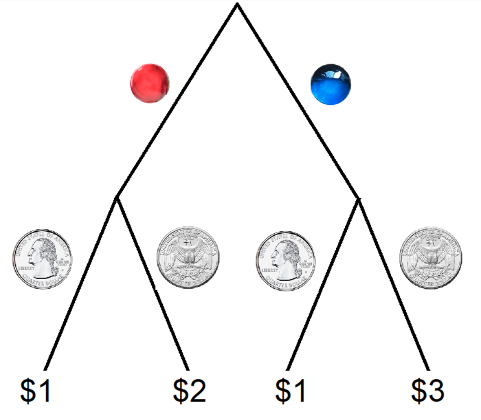


Savage: No probabilities are known

State space: $\Omega = \{RH, RT, BH, BT\}$. Outcomes: $X = \{\$1, \$2, \$3\}$

Act: $f(RH) = \$1$, $f(RT) = \$2$, $f(BH) = \$1$, $f(BT) = \$3$

An Example



Anscombe-Aumann (AA): Probabilities known *only* in the 2nd stage

State space: $\Omega = \{R, B\}$. Outcomes: $X = \{\$1, \$2, \$3\}$

AA-act: $f(R) = (\$1, 0.5; \$2, 0.5)$, $f(B) = (\$1, 0.5; \$3, 0.5)$

The vNM Framework: Objective Lotteries

vNM (1944)

- Outcomes: $x \in X = \{x_1, \ldots, x_n\}$
- Simple lotteries: $p \in L_1 = \Delta(X)$, $p = (x_1, p_1; \ldots; x_n, p_n)$
- Mixture operation:

$$\alpha p + (1-\alpha)q = (x_1, \alpha p_1 + (1-\alpha)q_1; \ldots; x_n, \alpha p_1 + (1-\alpha)q_1) \in L_1$$

- $\succeq$ over L_1 (complete, transitive, & continuous, so $\exists\, U(p)$)
- vNM's Mixture Independence Axiom (IND):

$$p \succeq q \iff \alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r$$

vNM EU Theorem: $\succeq$ satisfies Mixture Indepenndnece if and only if

$$\exists u(\cdot) : U(p) = \sum_x p(x)u(x)$$

We just need to learn your risk aversion ($u(x)$ "utility index")

Segal (1990): Compound Lotteries

- Two-stage lottery: $P = (q^1, P_1; \ldots; q^m; P_m) \in L_2$
- $\succeq$ now over L_2
- $\delta_q^1 = (q, 1) \in L_2$ and $\delta_q^2 = ((x_1, 1), q_1; \ldots; (x_n, 1), q_n) \in L_2$
- Time Neutrality Axiom: $\delta_q^1 \sim \delta_q^2$
- "$p \succeq q$" means $\delta_p^1 \succeq \delta_q^1$
- "$x \succeq y$" means $\delta_{(x,1)}^1 \succeq \delta_{(y,1)}^1$
- Reduced Lottery: given $P = (q^1, P_1; \ldots; q^m; P_m)$, let

$$r(P) = \left(x_1, \sum_{j=1}^{m} P_j q_1^j; \ldots; x_n, \sum_{j=1}^{m} P_j q_n^j \right) \in L_1$$

- ROCL Axiom: $P \sim r(P)$ (technically, $P \sim \delta_{r(P)}^1$)

Segal (1990): Compound Lotteries

- Recall Mixture operation for simple lotteries:

  $$\alpha p + (1-\alpha)q = (x_1, \alpha p_1 + (1-\alpha)q_1; \ldots; x_n, \alpha p_1 + (1-\alpha)q_1) \in L_1$$

  Mixture is "in between" p and q, in the same space (L_1)
- 1st-Stage Mixture operation:

  $$\alpha P + (1-\alpha)Q = (q^1, \alpha P_1 + (1-\alpha)Q_1; \ldots; q^n, \alpha P_1 + (1-\alpha)Q_1) \in L_2$$

- Compound operation:

  $$\alpha p \oplus (1-\alpha)q = (p, \alpha; q, 1-\alpha) \in L_2$$

- Compound Independence Axiom (1st attempt):

  $$p \succeq q \iff \alpha p \oplus (1-\alpha)r \succeq \alpha q \oplus (1-\alpha)r$$

- Problem: only applies to 2-element compound lotteries

Segal (1990): Compound Loteries

- Replacement operation: Fix $P = (q^1, P_1; \ldots; q_n, P_n)$. Define

$$[P|p, i] = (q^1, P_1; \ldots; p, P_i; \ldots; q_n, P_n)$$

  "Replace the ith branch with p"

- Compound Independence Axiom (fully general):

$$p \succeq q \iff [P|p, i] \succeq [P|q, i]$$

How does this apply to experiments?

Experiments as Compound Lotteries

Imagine two decisions, each a choice between two lotteries

1. $D_1 = \{p^1, q^1\}$
2. $D_2 = \{p^2, q^2\}$

Coin flip determines whether D_1 or D_2 is paid.
True preference: $p^1 \succ q^1$ and $p^2 \succ q^2$

- Would the subject want to lie in D_1?
- Fix choice of p^2 in D_2. Compare p^1 vs. q^1
- Announce p^1: get $\frac{1}{2}p^1 \oplus \frac{1}{2}p^2$
- Announce q^1: get $\frac{1}{2}q^1 \oplus \frac{1}{2}p^2$

Incentive compatibility: If $p^1 \succ q^1$ then

$$\frac{1}{2}p^1 \oplus \frac{1}{2}p^2 \succ \frac{1}{2}q^1 \oplus \frac{1}{2}p^2.$$

That's exactly Compound Independence!!

Segal (1990): Compound Lotteries

What would EU for two-stage lotteries look like?

$$U(P) = \sum_i P_i \left(\sum_x q^i(x)u(x) \right)$$

EU with ROCL?

What axioms give EU for two-stage lotteries? EU with reduction

Segal (1990): Compound Lotteries

When do we get EU with reduction?

1. Mixture independence $\Rightarrow$ EU on simple lotteries
   - But depends on which timing you use: δ^1 vs δ^2
2. MixIND + Time Neutrality $\Rightarrow$ 2nd stage EU regardless of δ^1 or δ^2
   - But might not have EU in 1st stage
3. MixIND + Time Neutrality + CompIND $\Rightarrow$ EU w/ Reduction
   - Compound Independence "connects" the two stages

What's the role of ROCL?

1. MixIND + TimeNeut + CompIND $\Rightarrow$ ROCL (see above)
2. ROCL connects the two IND axioms
   2.1 ROCL + MixIND $\Rightarrow$ CompIND
   2.2 ROCL + CompIND $\Rightarrow$ MixIND
   So you can replace either with ROCL and still get EU w/ Reduction

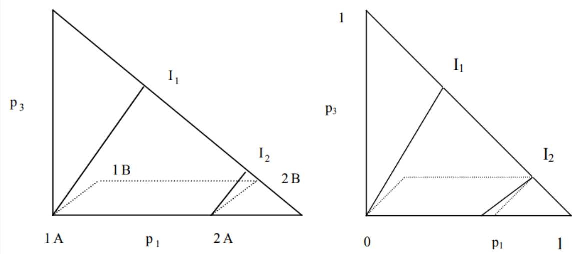
Allais Paradox

Why we might not want to assume MixIND (thus, EU):

- **Option 1A**: 100% chance of $1M
- Option 1B: 10% $5M, 89% $1, 1% $0

and

- Option 2A: 11% $1M, 89% $0
- **Option 2B**: 10% $5M, 90% $0



Need “fanning out” indifference curves (could be nonlinear)

Stochastic Dominance

What if we don't want to assume EU? Minimal: $\succeq$ respects dominance

- Second-Stage ("subjective") Stochastic Dominance:

$$p \sqsupset_2 q \iff \forall x \in X \sum_{y:y \succeq x} p(y) \geq \sum_{y:y \succeq x} q(y) \text{ (one strict)}$$

- First-Stage Stochastic Dominance:

$$P \sqsupset_1 Q \iff \forall p \in supp(P,Q) \sum_{q:q \succeq p} P(q) \geq \sum_{q:q \succeq p} Q(q) \text{ (one strict)}$$

- 2nd-Stage Monotonicity: $p \sqsupseteq_2 q \Rightarrow p \succeq q$ and $p \sqsupset_2 q \Rightarrow p \succ q$
- 1st-Stage Monotonicity: $P \sqsupseteq_1 Q \Rightarrow P \succeq Q$ and $P \sqsupset_1 Q \Rightarrow P \succ Q$
- CompIND+TimeNeut connects the two monotonicity axioms
    - (CompIND+TimeNeut) + 2nd-Stage MONO $\Rightarrow$ 1st-Stage MONO
    - (CompIND+TimeNeut) + 1st-Stage MONO $\Rightarrow$ 2nd-Stage MONO
    - Can replace (CompIND+TimeNeut) with ROCL

Back to Experiments

If we only have dominance axioms, can we say anything about experiments?

Recall: CompoundIND $\Rightarrow$ tell the truth

Lemma: 1st-Stage Monotonicity $\Rightarrow$ CompoundIND ($\Rightarrow$ tell the truth)

Proof

- Suppose $p \succeq q$
- Then $[P|p,i] \sqsupseteq_1 [P|q,i]$ (they only differ on the ith branch)
- By 1st-Stage MONO, $[P|p,i] \succeq [P|q,i]$

Back to Experiments

So any non-EU theory satisfying CompIND (or 1st-Stage MONO) is fine...

But recall:

$$\text{CompIND} + \text{ROCL} \Rightarrow \text{MixIND} \Rightarrow \text{EU (for either } \delta^1 \text{ or } \delta^2\text{)}$$

or, the contrapositive:

$$\text{non-EU theory} \Rightarrow \neg\,\text{MixIND} \Rightarrow (\neg\,\text{CompIND}) \text{ or } (\neg\,\text{ROCL})$$

So, if you have a non-EU theory, you either have:

1. $\neg$ CompIND, so IC will fail (for some experiments), or
2. $\neg$ ROCL, in which case you can still have CompIND & IC

So we better hope ROCL fails!! (Hint: it does. More later...)

Non-EU Theories

What non-EU theories are there in this framework?

1. Probability Weighting (from original Prospect Theory 1977)
    - $p = (x_1, p_1; \ldots; x_n, p_n)$

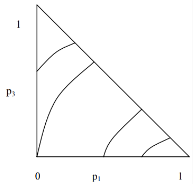
$$U(p) = \sum_{i=1}^{n} u(x_i) w(p_i)$$

    - Weighting function $w(\cdot)$ is concave then convex.
    - Problem! This can violate (2nd-Stage) MONO
    - Solution: Editing phase. People never pick dominated lotteries

2. Rank-Dependent Utility
   - $p = (x_1, p_1; \ldots; x_n, p_n)$ where $x_1 < x_2 < \cdot < x_n$

$$U(p) = \sum_{i=1}^{n} u(x_i) \underbrace{\left[w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j) \right]}_{\text{weighted “probability” of } x_i}$$

   - Weighting function $w(\cdot)$ is concave then convex. $p^* \approx 0.4$?
   - Weights on *cumulative* probability avoids MONO violations
   - Quiggen (1982), Prelec (1993) weighting function, many others

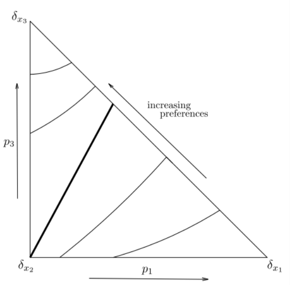


Non-EU Theories

3. Cautious Expected Utility (Cerreia-Vioglio et al. 2015)
    - Certainty equivalent of p solves $u(c_p^u) = \sum_x p(x)u(x)$, or $c_p^u = u^{-1}(\sum_x p(x)u(x))$
    - Agent has a set of utility indices $\mathcal{V}$ and is "pessimistic"

$$U(p) = \inf_{u \in \mathcal{V}} u^{-1}\Big(\sum_x p(x)u(x)\Big) = \inf_{u \in \mathcal{V}} c_p^u$$

    - NCI (weakening of MixIND): $p \succeq \delta_x \Rightarrow \alpha p + (1-\alpha)r \succeq \alpha\delta_x + (1-\alpha)r$



Other Non-EU Theories

4. Cumulative PT (K&T 1992): RDU with loss aversion
5. Weighted EU (Chew & McCrimmon 1979)
6. Decision Weighted Utility (Handa 1977), but violates MONO

Today: RDU and CPT are the most popular

The Savage Framework: Entirely Subjective Beliefs

Jimmy Savage

 Leonard "Jimmy" Savage

1. Eminem of statistics (genius from Detroit)
2. Wayne State $\rightarrow$ Michigan BS & PhD in math (1941)
3. IAS Princeton, then Chicago. Milton Friedman & W. Allen Wallis mentors
4. WWII: assistant to John von Neumann
5. *The Foundation of Statistics* (1954)
    - Subjective expected utility without objective lotteries

Savage (1954)

- States: $\omega \in \Omega$ (need Ω to be infinite)
- Events: $E \subseteq \Omega$
- Outcomes: $x \in X$ (prizes, consequences...)
- Acts: $f : \Omega \to X$. $\;f \in \mathcal{F} = X^{\Omega}$
- $\succeq$ over $\mathcal{F}$
- Notation: xEy is binary act where $f(E) = x$ and $f(E^c) = y$

Savage's omelette example: crack next egg into separate bowl?

- $\Omega = \{$good egg,rotten egg$\}$
- f = crack into same bowl. g = crack into separate bowl
- $f(good)$ = omelette, wash 1 bowl. $f(bad)$ = no omelette, wash 1
- $g(good)$ = omelette, wash 2 bowls. $g(bad)$ = omelette, wash 2
- $f \succeq g$ or $g \succeq f$?

Savage (1954)

What would EU look like??

- vNM EU: assume linearity, just need to learn $u(x)$
- Savage: assume more, but need to learn $u(x)$ **and** $p(\omega)$

The goal:

$$U(f) = \sum_{x \in X} p(\underbrace{\{\omega : f(\omega) = x\}}_{\text{Event ``}x\text{ is paid''}})\, u(x)$$

$$= \sum_{x \in X} p(\, f^{-1}(x)\,)\, u(x)$$

The building blocks: Savage's 6 Postulates (P1–P6) (ignore P7 here)
P1: $\succeq$ is complete, reflexive, and transitive ("ordering")
P5: There are $x, y \in X$ s.t. $x \succ y$ ("non-degeneracy")

The hard part: How to learn $p(\omega)$ from $\succeq$??

Ramsey (1926)

 Frank Plumpton Ramsey

- Born into academic privilege, Cambridge, England
- Easy-going, simple, modest, loved swimming
- Translated Wittgenstein, went to Austria, became his friend
- Math undergrad at Cambridge, advisor was Keynes. No PhD
- Keynes (1921) *A Treatise on Probability*: prob. must be objective
- Ramsey (1926) "Truth and Probability": probability is subjective
- Your beliefs are *defined by* the bets you'd make. The odds.
- Problem: confounded with risk aversion! Assume risk neutrality?
- De Finetti (1937) independently developed same ideas
- Liver problems, surgery, died at age 26. Infection from river?

Savage (1954)

How can we learn your beliefs?

- Would you rather bet on event E or F? Pick $1E0$ or $1F0$?
- Leads to a ranking of all possible events: $E \trianglerighteq F$
- Beliefs work like utility! $E \trianglerighteq F \Rightarrow p(E) \geq p(F)$.
- p is "qualitative probability". Ordinal, unless we add structure
- Can $p(E) = 0$? E is "null" if $fEh \sim gEh$ regardless of f, g, h

1. The stakes of the bet shouldn't matter:
   P4: If $x' \succ x$ and $y' \succ y$ then $(x'Ex \succeq x'Fx) \iff (y'Ey \succeq y'Fy)$
2. "Small" events exist:
   P6: For any $f \succ g$ and x I can find "small enough" events A_1, A_2 such that $f \succ xA_1g$ and $xA_2f \succ g$
   (Substituting in x doesn't change the ordering of f and g)
3. Eventwise Monotonicity (similar to Compound Independence):
   P3: $x \succeq y \iff xEg \succeq yEg$ (if E is not null)

Savage (1954)

We're almost there!!

1. **P1: Ordering**
2. **P2: ???**
3. **P3: Eventwise Monotonicity** (dominance)
4. **P4: Weak Comparative Probability** (stakes don't matter)
5. **P5: Nondegeneracy**
6. **P6: Small Event Continuity**

The missing piece (P2) helps gives us linearity for EU

Hartmann (2020): P3 is implied by the others

Savage (1954)

P2: The Sure-Thing Principle

$$f'Eg \succeq fEg \Rightarrow f'Eh \succeq fEh$$

"I can rank f' vs f conditional on E,
and what's paid off of E won't matter."

Has a flavor of Mixture Independence from vNM

Savage's Subjective Expected Utility (SEU) Theorem:

$$\succeq \text{ satisfies P1–P6} \Rightarrow \exists u, p : \; U(f) = \sum_x p(\{\omega : f(\omega) = x\})u(x)$$

Which axiom is relevant for experiments?

Experiments as Acts

Two decisions, choice objects are abstract (acts, lotteries, $, ...)

1. $D_1 = \{a^1, b^1\}$
2. $D_2 = \{a^2, b^2\}$

Payment act: $\omega_1 \mapsto D_1$ is paid, and $\omega_2 \mapsto D_2$ is paid
True preference: $a^1 \succ b^1$ and $a^2 \succ b^2$

- Would the subject want to lie in D_1?
- Fix choice of a^2 in D_2. Compare a^1 vs. b^1. Let $E = \{\omega_1\}$
- Announce a^1: get a^1Ea^2
- Announce b^1: get b^1Ea^2

Incentive compatibility: If $a^1 \succ b^1$ then

$$a^1Ea^2 \succ b^1Ea^2$$

That's **P3** (eventwise monotonicity)!! $x \succeq y \Rightarrow xEg \succeq yEg$

Finite States?

Could we have a finite number of states?

- Representation would not be unique
  - Small changes to p and u wouldn't alter $\succeq$
- Kraft, Pratt, & Seidenberg (1959)
  - Can construct $\trianglerighteq$ that is not represented by any probability measure p!
  - Requires $n = 5$, not very intuitive (to me)
- Gul (1992) gets SEU for finite Ω
  - Requires $\exists E$ where $E \sim_{\trianglerighteq} E^C$
  - Other, stronger axioms
- Others have, too (Wakker, Davidson & Suppes, Nakamura...)

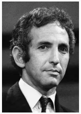
The Paradoxes

- OK, we care about P3 (eventwise montonicity)
- P3 is part of EU
- But we know EU is violated!
- Allais paradox: indifference curves aren't parallel
- Even worse: Ellsberg paradox
    - People don't even *have* well-defined probabilities!

Which axioms do these paradoxes violate??

Is P3 (eventwise monotonicity) okay???

Ellsberg (1961)

 Daniel Ellsberg

- Detroit kid, Marine, RAND Corp
- Harvard PhD student, wrote the Ellsberg paradox
- Worked at the Pentagon under McNamara, went to Vietnam
- Left Pentagon for RAND. Contributed to the “Pentagon Papers”, a complete analysis of the conduct of the US military in Vietnam
- Became sympathetic to war resisters
- Leaked the Pentagon Papers to the NY Times
- Politically embarrassing to Kennedy, Johnson, Nixon
- Nixon’s “White House Plumbers” and Watergate ensued
- Tried for espionage, acquitted
- Died June 16, 2023

Ellsberg (1961)

Urn: 30 red + (60 black or yellow)

	30 balls	60 balls	
	red	black	yellow
f_1 (Bet R)	\$100	\$0	\$0
f_2 (Bet B)	\$0	\$100	\$0
g_1 (Bet RY)	\$100	\$0	\$100
g_2 (Bet BY)	\$0	\$100	\$100

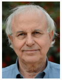
- People avoid ambiguity: $f_1 \succ f_2$ but $g_2 \succ g_1$
- Clearly violates P2 Sure Thing Principle
- Also means you can't have a probability!
    - $f_1 \succ f_2 \Rightarrow R \rhd B$
    - $g_2 \succ g_1 \Rightarrow BY \rhd RY \Rightarrow B \rhd R$
- Not a test of P3. Phew!
- What theories can accommodate this "ambiguity aversion"?

Frank Knight

 Frank Knight

- Another Chicago school patriarch
- Born 1885. Tennessee undergrad, Cornell PhD 1916
- Advisor of Milton Friedman, George Stigler
- Praised by Coase, Hayek, Samuelson.
- Known for *Risk, Uncertainty, and Profit* (1921)
  - Based on his PhD work at Cornell
- Distinguished “risk” and “uncertainty” (ambiguity)
- Agrued that they differ in a “deep” way
- Claimed that uncertainty can lead to positive profits in competitive industries
- Today: Ambiguity = “Knightian uncertainty”

David Schmeidler

 David Schmeidler

- Born in Poland, 1939. Family evaded WWII in Russia
- Studied math at Hebrew U., PhD under Robert Aumann
- Full-time at Tel Aviv U. since 1971, part-time at OSU since 1987
- Father of “post-Savage” decision theory, incorporating ambiguity/Knightian uncertainty
- Died March 17, 2022

Machina & Schmeidler (1992):
What does it mean to “have a probability” if you’re not EU??
(Note: we’re forgetting Ellsberg’s paradox for now...)

Machina & Schmeidler (1992)

What we want:

1. You have $\trianglerighteq$ over events that leads to $p(\omega)$ (P1,P4,P5,P6)
2. Acts "become" lotteries via $p(\omega)$
3. You respect FOSD over those lotteries (P3)

Can we just drop P2?? Not quite!
P4 is what really gives $\trianglerighteq$, but only for two outcomes (E and E^c)
Need to strengthen P4 to deal with more outcomes

Let $xEyFg$ be "x on E, y on F, and g otherwise"

P4*: $(\forall E, F$ disjoint$)(\forall x' \succ x, y' \succ y)\ (\forall g, h)$

$$x'ExFg \succeq xEx'Fg \Rightarrow y'EyFh \succeq yEy'Fh$$

"$E \trianglerighteq F$ regardless of stakes *or* what's paid outside of $E \cup F$"

Let $r(f, p)$ be the lottery generated by f using beliefs p

Theorem: If $\succeq$ satisfies P1,   , P3, P4*, P5, P6 then there exists

1. a subjective probability measure $p(\omega)$ on Ω
2. a function $V(p)$ over lotteries that respects FOSD

such that

$$U(f) = V(r(f, p))$$

They say $\succeq$ is *probabilistically sophisticated*

Ellsberg is a test of probabilistic sophistication?? Which axiom?

Allais vs. Ellsberg

ALLAIS

	1%	10%	89%
	#1	#2–11	#12–100
✓ f_1	\$100	\$100	\$100
f_2	\$0	\$500	\$100
g_1	\$100	\$100	\$0
✓ g_2	\$0	\$500	\$0

ELLSBERG

	30	––60––	
	red	black	yellow
✓ f_1	\$100	\$0	\$0
f_2	\$0	\$100	\$0
g_1	\$100	\$0	\$100
✓ g_2	\$0	\$100	\$100

EU: **P2**: $f'Eg \succeq fEg \Rightarrow f'Eh \succeq fEh$

Violated! $E = \{1, 2\text{–}11\}$

Violated! $E = \{\text{red,black}\}$

P4: $(x'Ex \succeq x'Fx) \iff (y'Ey \succeq y'Fy)$

Not tested. No E vs. F

Not tested!

P.S.: **P4***: $(x'ExFg \succeq xEx'Fg) \iff (y'EyFh \succeq yEy'Fh)$

Also not tested.

Violated! $E = \{\text{red}\}$, $F = \{\text{black}\}$

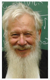
Ambiguity Aversion Literature

- Want to explain ambiguity aversion! ("Knightian uncertainty")
- So, models that violate probabilistic sophiciation.
- Specifically, violate P4*
- But, for experiments, we hope P3 is maintained!

I'll review these... but they mostly use a simpler framework!

A framework called...

The Anscombe-Aumann Framework

 Robert Aumann

- Born in Germany 1938, family fled to NYC before Kristallnacht
- City College of NY, math PhD at MIT 1955
- Knot theory. Loved puzzles. “Absolutely useless.” But, DNA...
- Learned game theory at Princeton, postdoc & sabbatical
- Hebrew U. Jerusalem since 1956, Stony Brook visitor
- Hugely important papers in many areas
    - Correlated equilibrium, common knowledge, division problems, continuum economies, epistemics, repeated games, cooperative game theory, integrals of correspondences...
- Aumann $\rightarrow$ Ehud Lehrer $\rightarrow$ Yaron Azrieli

 Frank Anscombe

- English statistician, 20yrs older than Aumann. Princeton
- Gave a lecture on Savage, which Aumann attended...

Anscombe-Aumann (1963)

- The Savage framework is..
  1. great because it gives us subjective $p(\omega)$ from $\succeq$
  2. too intractible to work with!
- Insight: vNM theorem works for *any* convex space

Theorem: Let K be a convex space, and $p, q \in K$ (not necessarily lotteries). Under vNM axioms (ordering, continuity, MixIND) $\exists U$:

$$U(\alpha p + (1-\alpha) r) = \alpha U(p) + (1-\alpha) U(r) \quad (U \text{ is affine})$$

Let K be a space of "acts that pay lotteries"
"Horse races (subjective) and roulette wheels (objective)"

$\Omega = \{\omega_1, \ldots, \omega_n\}$, $f = (p^1, \ldots, p^n)$
$f \in \mathcal{F} = \Delta(X)^\Omega$ is a convex space!

$$U(\alpha f + (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g)$$

How does this mixture operation work?

- At each ω, $f(\omega)$ and $g(\omega)$ are lotteries.
- $(\alpha f + (1-\alpha)g)(\omega) = \alpha f(\omega) + (1-\alpha)g(\omega)$
- Mixture lottery state-by-state

What does affine U imply? There exists *state-dependent* $u(x|\omega)$
vNM's **A1–A3** applied to $\Delta(X)^\Omega \Rightarrow$

$$U(f) = \sum_\omega \sum_x \underbrace{f(\omega)(x)}_{Pr(x) \text{ at } \omega} u(x|\omega)$$

The "weight" on state ω is embedded in $u(x|\omega)$. No $p(\omega)$.

Want state-independent utility index for EU, thus a separate $p(\omega)$

AA (1963)

A1–A3 give:

$$U(f) = \sum_\omega \sum_x \underbrace{f(\omega)(x)}_{Pr(x) \text{ at } \omega} u(x|\omega)$$

A4: CompIND: $p \succeq q \Rightarrow [f|p, i] \succeq [f|q, i]$
A4: Monotonicity: $(\forall\, \omega)\ f(\omega) \succeq g(\omega) \Rightarrow f \succeq g$ (HW: CompInd=MONO)
A5: Non-degeneracy: $(\exists f, g)\ :\ f \succ g$

"State independence" gives $u(x|\omega) = \alpha(\omega)u(x)$
$u(x)$ captures the curvature of $u(x|\omega)$, $\alpha(\omega)$ captures its "height"

Normalize these $\alpha(\omega)$'s to sum to one. Now it's a belief over Ω!!
AA EU Theorem: If $\succeq$ satisfies Ordering, Continuity, MixIND in Objective Lotteries, State Ind, and Non-Degeneracy then $\exists p, u$:

$$U(f) = \sum_\omega \underbrace{p(\omega)}_{\text{Subj. belief}} \sum_x \underbrace{f(\omega)(x)}_{Pr(x) \text{ at } \omega} u(x)$$

Back to Machina-Schmeidler

What if we want to relax EU? But maintain dominance and $\exists p$

- Machina Schmeidler (1995): Prob. Sophistication in AA world
  1. Replace A3 (MixIND) with FOSD Monotonicity
  2. Replace A4 with an axiom that only applies to two-outcome bets
     - Thus, affects $p(\omega)$ but doesn't restrict risk prefs.

A4*: for disjoint E and F,

$$1E0F0 \sim (1,\alpha)E(1,\alpha)F0 \Rightarrow pEqFr \sim (\alpha p+(1-\alpha)q)E(\alpha p+(1-\alpha)q)Fr$$

Indifferent between bet on E and lower-stakes bet on $E \cup F$
$\Rightarrow$ same indiff when payoffs are replaced by lotteries

Allows for $p(E) = \alpha p(E \cup F)$, regardless of "stakes"

A1,A2,FOSD-MONO,A4* $\Rightarrow$
$\exists p, V : f \succeq g \iff V(r(f,p)) \geq V(r(g,p))$

Non-EU Theories

What are the famous non-EU theories to explain Ellsberg, etc?

1. Schmeidler: Choquet Expected Utility
2. Gilboa-Schmeidler: Maxmin Expected Utility
3. KMM: Smooth Ambiguity Preference
4. Seo: Two-Stage EU
5. ⋮

But first, what exactly *is* ambiguity aversion??

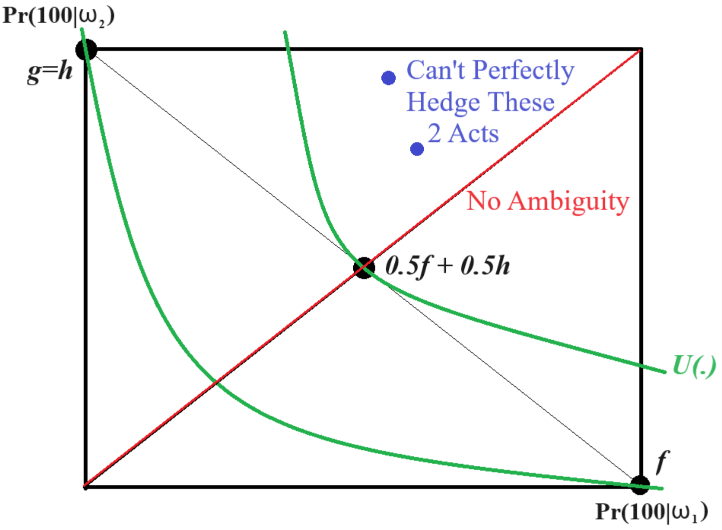
More than just violating P4*. There's a direction to it...

Schmeidler's Definition of Ambiguity Aversion

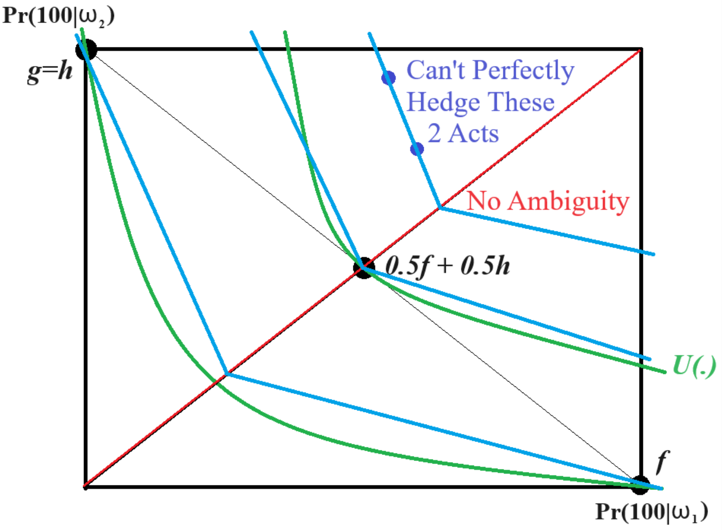
An ambiguity averse person should recognize "hedging" opportunities

- AA setting. Set $\Omega = \{\omega_1, \omega_2\}$
- $f = ((\$100, 1), (\$0, 1))$ (a bet on ω_1)
- $g = ((\$0, 1), (\$100, 1))$ (a bet on ω_2)
- Let $h = g$ (for use later)
- Suppose $f \sim g$, meaning $\{\omega_1\} \sim_{\trianglerighteq} \{\omega_2\}$
- $\frac{1}{2}f + \frac{1}{2}h = ((\$100, \frac{1}{2}; \$0, \frac{1}{2}), (\$100, \frac{1}{2}; \$0, \frac{1}{2}))$ pays same lottery in both states
- $\frac{1}{2}g + \frac{1}{2}h = g$, payoff still depends on the state
- First mixture "hedges away" ambiguity!
- $g \succeq f$ but $\frac{1}{2}f + \frac{1}{2}h \succ \frac{1}{2}g + \frac{1}{2}h$
- Violate AA's MixIND (thus, EU). Specifically, prefs are convex
- Convex preferences = "hedging incentive"

Schmeidler's Definition of Ambiguity Aversion



Schmeidler's Definition of Ambiguity Aversion



Kinked-linear indifference curves best capture hedging incentives

Choquet EU (Schmeidler 1989)

- Sort of like probability weighting...
- AA setting. $f = (p^1, p^2, \dots, p^n)$, each $p^i \in \Delta(X)$
- Goal:

$$U(f) = \sum_{\omega} v(\omega) \sum_{x} f(\omega)(x)\, u(x)$$

- But here $v(\cdot)$ can be non-additive: $v(A) + v(B) \neq v(A \cup B)$
  1. $v(\emptyset) = 0$,
  2. $v(\Omega) = 1$
  3. $A \subset B \Rightarrow v(A) \leq v(B)$
- Ambiguity aversion: "subadditive" $v(\cdot)$
  Technically, *convex*: $v(A) + v(B) - v(A \cap B) \leq v(A \cup B)$
- Example: Bent coin. $v(H) = 0.4$ and $v(T) = 0.4$
  - Normalize $u(1) = 1$, $u(0) = 0$
  - Bet on heads: $U(f) = 0.4 \cdot 1 \cdot u(1) + 0.4 \cdot 1 \cdot u(0) = 0.4$
  - Bet on tails: $U(g) = 0.4 \cdot 1 \cdot u(1) + 0.4 \cdot 1 \cdot u(0) = 0.4$
  - Bet on a fair coin: $U(h) = 1 \cdot (0.5\, u(1) + 0.5\, u(0)) = 0.5$
    (Recall $v(H \cup T) = 1$)

Choquet EU (Schmeidler 1989)

The problem with standard (Reimann) integration:

- Suppose v is non-additive (eg, convex)
- $(1, E_1; 1, E_2; 0, E_3) = (1, E_1 \cup E_2; 0, E_3)$ are identical
- Weight on winning event: $v(E_1) + v(E_2) \neq v(E_1 \cup E_2)$. Different!

Schmeidler's solution: use the Choquet (1955) integral!

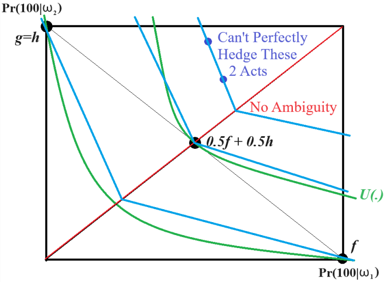
- Suppose $f = (x_1, E_1; \ldots, x_n, E_n)$, where $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$
- Reimann: $\sum_i x_i \, v(E_i)$
- Choquet: $\sum_i x_i \left[v(\cup_{j=1}^{i} E_j) - v(\cup_{j=1}^{i-1} E_j) \right]$
- Similar to RDU! Also avoids dominance violations

Above example:

$1\,[v(E_1)] + 1\,[v(E_1 \cup E_2) - v(E_1)] + 0\,[v(E_1 \cup E_2 \cup E_3) - v(E_1 \cup E_2)]$

$= 1\,[v(E_1 \cup E_2)] + 0\,[v(E_1 \cup E_2 \cup E_3) - v(E_1 \cup E_2)]$

Choquet EU (Schmeidler 1989)



Choquet calculation implies linear between two blue dots:

$$U(a) = 0.95\,[v(\omega_2)] + 0.55\,[v(\Omega) - v(\omega_2)]$$
$$U(b) = 0.75\,[v(\omega_2)] + 0.60\,[v(\Omega) - v(\omega_2)]$$
$$U(\frac{1}{2}a + \frac{1}{2}b) = 0.85\,[v(\omega_2)] + 0.575\,[v(\Omega) - v(\omega_2)] = \frac{1}{2}U(a) + \frac{1}{2}U(b)$$

Linear on either side of the 45-degree line!

Choquet EU (Schmeidler 1989)

Points on the same side of the 45-degree line are "comonotonic"

$$f(\omega) \geq f(\omega') \Rightarrow g(\omega) \geq g(\omega')$$

Change MixIND to **A3*:** Comonotonic Independence.
MixIND but only for comonotonic acts f and g
(Constant acts are comonotonic, so we do get EU over lotteries)

Theorem: A1, A2, ComonotonicIND, A4, A5 $\Rightarrow$
$\exists$ non-additive $v(\cdot)$ and $\exists$ $u(\cdot)$ unique up to p.a.t. s.t.

$$U(f) = \sum_i \underbrace{\left(v(\cup_{j=1}^{i}\omega_j) - v(\cup_{j=1}^{i-1}\omega_j) \right)}_{\text{Non-additive "belief" of } \omega_i} \underbrace{\sum_x f(\omega_i)(x)\, u(x)}_{\text{EU over lottery } f(\omega_i)}$$

Can have ambiguity-loving if $v(\cdot)$ is "superadditive"
Still satisfies Mono/CompIND (**A4**)! Good news for experiments
His student Itzhak Gilboa extended this to the Savage framework

Choquet EU becomes Maxmin EU

An alternative representation

- Suppose you have Choquet EU with ambiguity aversion
- Ambiguity aversion $\Rightarrow v(\cdot)$ is convex
  - $v(A) + v(B) - v(A \cap B) \leq v(A \cup B)$
- Convex $v(\cdot)$ has a "core" of additive distributions...
  - $C_v = \{p \in \Delta(\Omega) \ : \ (\forall E \subseteq \Omega) \ p(E) \geq v(E)\}$
  - Ex: if $v(H) = v(T) = 0.4$ then
    $C_v = \{p \in \Delta(\{H, T\}) : p(H) \in [0.4, 0.6]\}$
- ... the Choquet expectation can equivalently be written as

$$\sum_i \left[v(\cup_{j=1}^{i} \omega_j) - v(\cup_{j=1}^{i-1} \omega_j) \right] \left(\sum_x f(\omega_i)(x) \, u(x) \right)$$
$$= \min_{p \in C_v} \sum_\omega p(\omega) \left(\sum_x f(\omega)(x) \, u(x) \right)$$

- Convex $v(\cdot) \Rightarrow$ "Maxmin expected utility (MEU)"

Gilboa & Schmeidler (1989) Maxmin Expected Utility (MEU)

- **A1**: Ordering (complete & transitive)
- **A2**: Continuity
- **A3.1**: Certainty Independence: Let $h_c = (q, q, \ldots, q)$ be any constant act

  $$f \succ g \iff \alpha f + (1-\alpha)h_c \succ \alpha g + (1-\alpha)h_c$$

  (This gives kinked-linear indifference curves)
- **A3.2**: Uncertainty Aversion: $f \sim g \Rightarrow \alpha f + (1-\alpha)g \succeq f$ (Convex indifference curves)
- **A4**: Monotonicity / Compound Independence
- **A5**: Non-degeneracy

$\Rightarrow \exists$ closed, convex set of beliefs C and u s.t.

$$U(f) = \min_{p \in C} \sum_{\omega} p(\omega) \left(\sum_{x} f(\omega)(x)\, u(x) \right)$$

Choquet EU vs. MEU?

Choquet EU ($v(\cdot)$) to MEU (C)?

- If $v(\cdot)$ is convex (ambiguity averse) then C is core of $v(\cdot)$ ✓
- If $v(\cdot)$ is additive then core is just $v(\cdot)$, so CEU=MEU=EU
- If $v(\cdot)$ is amb. loving then $\nexists$ MEU representation
    - Max-EU representation??
- Aribtrary $v(\cdot)$??

MEU (C) to Choquet EU ($v(\cdot)$)?

- Given C, you can define $v(E) = \min_{p \in C} p(E)$, but...
    1. $v(\cdot)$ may not be convex, and even if it is...
    2. its core C_v may be different from the C you started with

I don't fully understand the differences :)

α-Maxmin

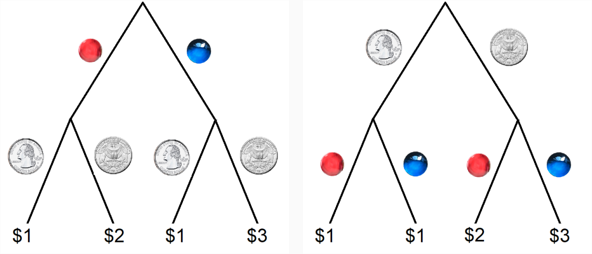
α-Maxmin EU is a generalization:

$$U(f) = \alpha \min_{p \in C} \sum_{\omega} p(\omega) \left(\sum_{x} f(\omega)(x)\, u(x) \right) + (1 - \alpha) \max_{p \in C} \sum_{\omega} p(\omega) \left(\sum_{x} f(\omega)(x)\, u(x) \right)$$

Things I haven't figured out yet:

- Who invented this model?
- What do indifference curves look like?
- Does an axiomatization exist?

Seo (2009) and Order Reversal



What if we flipped the order? Would it matter?

Modeling problem: How to put these in the same framework?

Solution: Lotteries over acts over lotteries!
Actually used in the original AA paper

Seo (2009) and Order Reversal

Let $P = (f^1, P_1; \ldots, f^n, P_n)$ be a (first-stage) lottery over acts.

First-stage mixture operation ("in between" P and Q):

$$\alpha P \oplus (1-\alpha)Q = (f^1, \alpha P_1 + (1-\alpha)Q_1; \ldots, f^n, \alpha P_n + (1-\alpha)Q_n)$$

If first stage is degenerate, acts like a compounding mixture:

$$\alpha \delta_f \oplus (1-\alpha)\delta_g = (f, \alpha; g, 1-\alpha)$$

First-Stage IND:

$$P \succeq Q \Rightarrow \alpha P \oplus (1-\alpha)R \succeq \alpha Q \oplus (1-\alpha)R$$

Seo (2009) and Order Reversal

Let $f = (p^1, \omega_1; \ldots; p^n, \omega_n)$ and $g = (q^1, \omega_1; \ldots; q^n, \omega_n)$

Second-stage mixture operation (MixIND at each ω):

$$\alpha\delta_f + (1-\alpha)\delta_g = (\alpha p^1 + (1-\alpha)q^1, \omega_1; \ldots; \alpha p^n + (1-\alpha)q^n)$$

Apply to degenerate f,g to get Third-Stage IND (vNM):

$$p \succeq q \Rightarrow \alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r$$

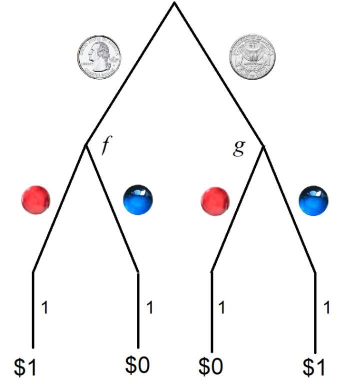
(technically, should write δ_{δ_p}, δ_{δ_q}, and δ_{δ_r} for p, q, and r)

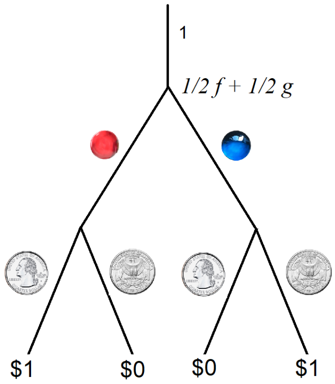
Seo (2009) and Order Reversal

Order Reversal Example:

Bet on red: $f(\text{red}) = (\$1, 1), f(\text{blue}) = (\$0, 1)$

Bet on blue: $g(\text{red}) = (\$0, 1), g(\text{blue}) = (\$1, 1)$





$$\alpha\delta_f \oplus (1-\alpha)\delta_g \quad \sim \quad \alpha\delta_f + (1-\alpha)\delta_g$$

Seo (2009) and Order Reversal

Order Reversal: mixing f and g “up” or “down” doesn’t matter.

Recall first-stage mixing for degenerate first stage (coin before):

$$\alpha\delta_f \oplus (1-\alpha)\delta_g = (f, \alpha; g, 1-\alpha)$$

And second-stage mixing for degenerate first stage (coin after):

$$\alpha\delta_f + (1-\alpha)\delta_g = (\alpha p^1 + (1-\alpha)q^1, \omega_1; \ldots; \alpha p^n + (1-\alpha)q^n)$$

Order Reversal:

$$\alpha\delta_f \oplus (1-\alpha)\delta_g \sim \alpha\delta_f + (1-\alpha)\delta_g$$

(AA 1963 actually had Order Reversal, but it’s since been simplified)

Seo (2009) and Order Reversal

The AA Axioms in this 3-stage framework:

- **A1:** Ordering
- **A2:** Continuity
- **A3.1:** First-Stage Independence
- **A3.2:** Order Reversal
- **A3.3:** Third-Stage Independence
- **A4:** Second-Stage Monotonicity

$$p \succeq q \Rightarrow [f|p, i] \succeq [f|q, i]$$

Theorem: A1–A4 imply $\exists$ additive belief p and utility index u s.t

$$U(P) = \sum_i P_i \sum_\omega p(\omega) \sum_x f^i(\omega)(x)\, u(x)$$

Seo (2009) and Order Reversal

Which axiom is important for experiments that pay one randomly?

- Monotonicity is in second stage (acts), so no.
- We need 1st-Stage IND for degenerate acts!
    - $\alpha\delta_f \oplus (1-\alpha)\delta_g = (f, \alpha; g, 1-\alpha)$

Fact: Order Reversal "connects" the two lottery stages:

- O.R. + 1st-Stage IND $\Rightarrow$ 3rd-Stage IND
- O.R. + 3rd-Stage IND $\Rightarrow$ 1st-Stage IND

Order Reversal and Incentive Compatibility

But that can be a problem!
Remember for 2-stage lotteries we had:

$$\text{CompIND} + \text{ROCL} \Rightarrow \text{MixIND} \Rightarrow \text{EU}$$

or

$$\text{non-EU theory} \Rightarrow \neg\, \text{MixIND} \Rightarrow (\neg\, \text{CompIND}) \text{ or } (\neg\, \text{ROCL})$$

$$\text{non-EU theory} \Rightarrow \neg\, \text{MixIND} \Rightarrow (\neg\, \text{I.C.}) \text{ or } (\neg\, \text{ROCL})$$

In this AA framework we have:

$$\text{1stStageIND} + \text{OR} \Rightarrow \text{3rdStageIND} \Rightarrow \text{EU}$$

or

$$\text{non-EU theory} \Rightarrow \neg\, \text{3rdStageIND} \Rightarrow (\neg\, \text{1stStageIND}) \text{ or } (\neg\, \text{OR})$$

$$\text{non-EU theory} \Rightarrow \neg\, \text{3rdStageIND} \Rightarrow (\neg\, \text{I.C.}) \text{ or } (\neg\, \text{OR})$$

If you want to allow for non-EU preferences, we better hope that ROCL or OR are not satisfied!

Seo (2009): What if we get rid of OR but keep both IND axioms?

Needs to modify Monotonicity to apply to first stage instead.
(AA didn't need this because they had O.R. to do it for them)

Let $r(P, q)$ be the reduced (two-stage) "objective" lottery generated by applying belief $q(\omega)$ over Ω. So, combine stages 2 and 3.
Note: $\succeq$ ranks 2-stage lotteries via degenerate 2nd stage (acts)

1st-Stage Dominance: $P \sqsupseteq Q$ iff $r(P, q) \succeq r(Q, q)$ for *every* $q \in \Delta(\Omega)$

A4*: If $P \sqsupseteq Q$ then $P \succeq Q$

Question: How does this compare to 1st-Stage IND?

Seo (2009) and SOSEU

Seo's axioms:

- **A1:** Ordering
- **A2:** Continuity
- **A3.1:** First-Stage Independence
- ~~**A3.2:** Order Reversal~~
- **A3.3:** Third-Stage Independence
- **A4*:** First-Stage Monotonicity

Theorem: A1–A4* imply $\exists$ a *belief over beliefs* $\pi \in \Delta(\Delta(\Omega))$, utility u, and a bounded, increasing function ϕ:

$$U(P) = \sum_i P_i \sum_{q \in \Delta(S)} \underbrace{\pi(q)}_{\Pr(\text{belief}=q)} \quad \underbrace{\phi(\sum_\omega q(\omega) \sum_x f^i(\omega)(x)\, u(x))}_{\text{SEU of } f \text{ w/ belief } q}$$

Ambiguity averse $\iff$ ϕ is concave

Seo (2009) SOSEU & Hedging

	30 marbles	60 marbles	
	R	B	Y
f	0	100	0
g	0	0	100

New state space: # black marbles: $\Omega = \{0, 1, 2, \ldots, 60\}$

$f(\omega) = (\frac{\omega}{90}, 100; \frac{90-\omega}{90}, 0) \quad g(\omega) = (\frac{60-\omega}{90}, 100; \frac{90-(60-\omega)}{90}, 0)$

$(\frac{1}{2}f + \frac{1}{2}g)(\omega) = (\frac{1}{3}, 100; \frac{2}{3}, 0) \;\; \forall\omega$

Suppose $q^1(20) = 1$ and $q^2(40) = 1$, with $\pi(q^1) = \pi(q^2) = 1/2$

Normalize $u(100) = 1$, $u(0) = 0$

EU of f at q^1: 20/90 EU of g at q^1: 40/90

EU of f at q^2: 40/90 EU of g at q^2: 20/90

$U(f) = \frac{1}{2}\phi(2/9) + \frac{1}{2}\phi(4/9) \quad U(g) = \frac{1}{2}\phi(4/9) + \frac{1}{2}\phi(2/9)$

EU of $0.5f + 0.5g$ at q^1 or q^2: $\frac{1}{2}\frac{20}{90} + \frac{1}{2}\frac{40}{90} = \frac{1}{3}$

$U(0.5f + 0.5g) = \frac{1}{2}\phi(1/3) + \frac{1}{2}\phi(1/3) = \phi(1/3)$

Concave $\phi \Rightarrow \phi(1/3) > \frac{1}{2}\phi(4/9) + \frac{1}{2}\phi(2/9)$

Convex preferences / preference for hedging

Smooth Ambiguity

Klibanoff, Marinacci & Mukerji (KMM, 2005)
"Smooth ambiguity" has basically the same form, but in a framework without "time"

Frameworks with multiple sources but without time?

Savage:

	ψ_1	ψ_2
ξ_1	ω_1	ω_2
ξ_2	ω_3	ω_4

AA: Can have $\succeq$ over $\mathcal{F} \cup \Delta(X)$
...but then you'll need different axioms for $f \succeq g$, $p \succeq q$, and $f \succeq p$

Literature on "source dependence"

Time and Risk

Discounted expected utility for a stream of lotteries:

$$U(p^0, p^1, p^2, \ldots) = \sum_{t=0}^{\infty} \delta^t \Big(\sum_x p^t(x)\, u(x)\Big)$$

Problem: *u* represents both risk preferences *and* time preferences!

Models that separate them:

- Kreps-Porteus (1978)
- Chew-Epstein (1989)
- Epstein-Zin (1989)

DeJarnette et al. (2020): If payment date is uncertain ("time lottery") then DEU predicts risk-seeking preferences! But experiment shows risk aversion. They provide new generalizations of DEU.

That's It!

- That's the end of this primer
- The DT literature is huge, and worth exploring.
- But sometimes axioms feel like framing effects
    - Example: Order Reversal
    - And, can we really control which order subjects perceive?
- ...so things get incredibly nuanced
- ...which unfortunately leads to a lot of fights

Regardless, this lays the foundation for formulating a theory of incentives in experiments