Designing Stable Mechanisms for Economic Environments

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Behavioral  Mechanism Design  Motivation
Objective: Design a game so that agents reach some desired objective in equilibrium
1. Starting point: Groves & Ledyard 1977
   1a. Nash implementation
   1b. ‘Economic’ Environments:
       Continuity, complexity (message space size), etc.
       Differentiability
2. Lesson from experiments: Stability matters
   • Chen & Plott 1996: ‘stability’ matters
   • Chen & Tang 1998: supermodularity
   • Arifovic & Ledyard 2003: something weaker
   • Healy 2006: dominant diagonal? specific dynamic?
   • Arifovic & Ledyard 2008: even weaker...
   • Current state of knowledge: supermodularity is sufficient.
This Paper

1. Understand how to develop G-L-like mechs.
2. Add ‘stability’ to the design constraints.

- Economic Environment: Two commodities
  \[ x_i = \text{numéraire}, \quad y_i = \text{private or public good} \]
- SCC: Walrasian or Lindahl equilibria (Hurwicz ’79)
- Continuously diff’bl mechanisms with ‘small’ strategy spaces

**Theorem 1:** Green-Laffont-type necessary cond’n:
\[ \text{tax}_i(m) = \text{price}_i(m - i) y_i(m) \]

**Theorem 2:** Impossibility results for 1-dimensional \( m \):
- WE: No mechanism. LE: No ‘stable’ mechanism.

**Theorem 3:** Convert any mechanism into a stable mechanism by adding a dimension to \( \mathcal{M} \)
The Economic Environment

- Agents: $i \in \{1, 2, \ldots, n\}$.
- Work with net trades; no consumption set boundaries
- Agent $i$’s endowment: $\omega_i = (0, 0)$.
- Net trade vector $z_i = (x_i, y_i)$
  - $x_i \in \mathbb{R}$: numeraire good
  - $y_i \in \mathbb{R}$: non-numeraire good (pub. or pvt)
- Agent $i$’s type: $\theta_i \in \Theta_i$ (complete information.)
- Later: QSL Preferences: $v_i(y_i|\theta_i) + x_i$.
  - $v_i$ is differentiable, strictly concave.
A Walrasian equilibrium is \((z^*, p^*)\) such that

1. \(\sum_i z^*_i = 0\),
2. each \(z^*_i\) maximizes \(u_i\) s.t. \(x_i + p^* y_i \leq 0\).

Public good: Set \(c(y) = \kappa y\).

A Lindahl equilibrium is \((z^*, p_1^*, \ldots, p_n^*)\) such that

1. \(\sum_i x^*_i + \kappa y^* = 0\),
2. each \(z^*_i\) maximizes \(u_i\) s.t. \(x_i + p^*_i y_i \leq 0\), and
3. \((\sum_i p^*_i) y - \kappa y\) is maximized at \(y^*\).

Walrasian and Lindahl correspondences: \(f : \Theta \rightarrow Z\)
Mechanisms

- Real-message mechanisms:
  - Strategy space: $M_i = \mathbb{R}^{K_i} \ \forall i$
  - Outcome functions: $(y_i(m), x_i(m))_i$

- Given a mechanism $(M, h)$, the Nash correspondence $\nu : \Theta \rightarrow M$ identifies the set of Nash equilibria for each $\theta$.

- A mechanism $(M, h)$ implements a social choice correspondence if $h(\nu(\theta)) = f(\theta)$ for all $\theta$. 
Supermodularity & Stability

Previous literature: supermodularity $\Rightarrow$ stability.

Supermodularity:

1. $\frac{\partial^2 u_i}{\partial m_{ik} \partial m_{il}} \geq 0$ for all $i, k \neq l$.
2. $\frac{\partial^2 u_i}{\partial m_{ik} \partial m_{jl}} \geq 0$ for all $i \neq j, k, l$.
3. Strategy space is a closed interval.

Milgrom & Roberts: ‘adaptive dynamics’ converge to $[\text{NE}, \overline{\text{NE}}]$

First 2 conditions: increasing BR curves.

Last condition: ignored in mechanism design!! Problem??
The Power of Supermodularity

Both games are “supermodular”. Left game is stable, right is not. Slope of BR curves matters!
The Power of Supermodularity

Unstable game: boundaries create ‘bad’ (stable) corner equilibria. ‘Stability’ property of supermodularity vacuous here.

\[ BR_i(m_j) = \frac{1}{2} m_j \quad \text{and} \quad BR_i(m_j) = 2m_j \]
“Counter-Example” Mechanism

Assume $v_i''(\theta_i) \in [-M, 0)$ for all $\theta \in \Theta$. Choose

$$y(m) = \sum_{i=1}^{n/2} m_i - \sum_{n/2+1}^{n} m_i$$

$$q_i(m) = \begin{cases} \frac{\kappa}{n} - \gamma \sum_{j \neq \{i, i + \frac{n}{2}\}} m_j & \text{if } i \leq n/2 \\ \frac{\kappa}{n} + \gamma \sum_{j \neq \{i, i + \frac{n}{2}\}} m_j & \text{if } i > n/2. \end{cases}$$

Supermodular if $\gamma > M$. But best response dynamic:

![Graph showing best response dynamic]
Van Essen’s suggestion:

- Can we make mechanisms with BR curves that are contraction mappings?
- \( \|BR(x) - BR(y)\| \leq \alpha \|x - y\| \) for \( \alpha \in (0, 1) \).
- For now, assume BR is single-valued.

**Definition**

A mechanism is **contractive on** \( \Theta \) if \( BR \) is non-empty, closed, and bounded, and for every \( \theta \in \Theta \) there exists some \( \alpha \in (0, 1) \) such that for every \( m, m' \in M \),

\[
\|BR(m') - BR(m)\| \leq \alpha \|m' - m\|.
\]
Does Contractive Imply Stable?

- $\overline{NE} = [m^*, \overline{m}^*]$
- Learning dynamic: $\{m(t)\} \subset M$
- Best response: $BR : M \to M$
- $H(t', t) = \left[ \min\{m(t'), \ldots, m(t)\}, \max\{m(t'), \ldots, m(t)\} \right]$
- $\overline{BR}(M') = \left[ \inf BR(M'), \sup BR(M') \right]$
- $\overline{BR}^*(M') = \left[ \min(\overline{BR}(M') \cup \overline{NE}), \max(\overline{BR}(M') \cup \overline{NE}) \right]$

Adaptive Best Response Dynamic:
$(\forall t')(\exists t'')(\forall t \geq t'') \ m(t) \in \overline{BR}^*(H(t', t - 1)).$

**Theorem:** If $\{m(t)\}$ is an ABR Dynamic and $BR(\cdot)$ is contractive then $m(t)$ converges to $\overline{NE}$. 
OK... how can we make a mechanism contractive?
Step 1: Understand how mechanisms look & feel.

**Trivial Observation:**

Every mechanism’s numeraire outcome functions can be written as

\[
x_i(m) = -q_i(m-i) y_i(m) - g_i(m).
\]

‘Price’ ‘Qty’ ‘Penalty’

Note: ‘Price-taking’ assumption
$m_i \in \mathbb{R}^1$

\[ y(m) = \sum_i m_i \]

\[ q_i(m_{-i}) = \frac{\kappa}{n} \]

\[ g_i(m) = \frac{\gamma}{2} \left[ \frac{n-1}{n} (m_i - \mu_{-i})^2 - \sigma_{-i}^2 \right] \]

\[ \mu_{-i} = \frac{1}{n-1} \sum_{j \neq i} m_j \quad \sigma_{-i}^2 = \frac{1}{n-2} \sum_{j \neq i} (m_j - \mu_{-i})^2 \]

Notes:

- Does not implement Lindahl allocations
- May not be individually rational (see Hurwicz 79)
- Free parameter $\gamma$ can guarantee stability*
\[ m_i \in \mathbb{R}^1 \]

\[ y(m) = \sum_i m_i \]

\[ q_i(m_{-i}) = \frac{k}{n} + m_{i+1} - m_{i+2} \]

\[ g_i(m) \equiv 0 \]

Notes:

- Implements Lindahl allocations (⇒ IR)
- Uses only one dimension
- Agents are ‘price-taking’
- “Wildly” unstable
\[ m_i = (s_i, z_i) \in \mathbb{R}^2 \]

\[ y_i(m) = s_i - \frac{1}{n-1} \sum_{j \neq i} s_j \]

\[ q_i(m_{-i}) = -\frac{1}{n-1} \sum_{j \neq i} z_j \]

\[ g_i(m) = (z_i - \frac{1}{n-1} \sum_{j \neq i} z_j)^2 + S_i(m_{-i}) \]

Notes:

- Implements Walrasian allocations using two dimensions
- \( g_i = 0 \) in equilibrium
- Agents are ‘price-taking’
- No free parameter; may not be stable
$m_i = (s_i, z_i) \in \mathbb{R}^2$

$y(m) = \sum_i s_i$

$q_i(m_{-i}) = \frac{\kappa}{n} - \sum_{j \neq i} s_j + \frac{1}{n} \sum_{j \neq i} z_j$

$g_i(m) = -\frac{1}{2} (z_i - y(m))^2$

Notes:

• Implements Lindahl using two dimensions
• $g_i = 0$ in equilibrium
• Agents are ‘price-taking’
• Globally stable (adjustment process) with QSL prefs
\[ m_i = (s_i, z_i) \in \mathbb{R}^2 \]

\[ y(m) = \sum_i s_i \]

\[ q_i(m_{-i}) = \frac{\kappa}{n} - \gamma \sum_{j \neq i} s_j + \frac{\gamma}{n} \sum_{j \neq i} z_j \]

\[ g_i(m) = -\frac{1}{2} (z_i - y(m))^2 + \frac{\delta}{2} \sum_{j \neq i} (z_j - y(m))^2 \]

Notes:

- Implements Lindahl using two dimensions
- \( g_i = 0 \) in equilibrium
- Agents are ‘price-taking’
- Induces supermodular* game with QSL prefs
Other mechanisms:

- de Trenqualye 1989
- Vega-Redondo 1989
- Kim 1996
- Corchon & Wilkie 1996

In all of these...

- Implement Lindahl using two dimensions
- \( g_i = 0 \) in equilibrium
- Agents are ‘price-taking’
Eerie Similarities

Why are these mechanisms so similar?

How do they work?

How much freedom is there to play with them?
The Graphical View

What you can achieve by changing $m_i$ (given $m_{-i}$)
The Local Price

\[ P_{ik}(m_{ik}'' \mid m_{-ik}) := \frac{-\partial x_i / \partial m_{ik}}{\partial y_i / \partial m_{ik}} \]

Slope of \( \chi_i \) is the ‘local price’.
Nash Equilibrium Points

Possible Nash equilibrium points given $u_i$ or $u_i'$. 
Possible Walrasian allocations given $u_i$ or $u_i'$. 
Triple tangency is necessary for NE outcome to be WE.
Rich-enough type space $\Rightarrow$ ANY $m$ is a NE.
‘Bad’ Nash Equilibria

But now the mechanism doesn’t implement Walrasian allocations!
The Necessary Condition

Only way to avoid ‘bad’ equilibria: \( t_i(m) = q_i(m_{-i})y_i(m) \).
Conclusion So Far

- If every strategy is an equilibrium, all equilibrium outcomes need to be on a budget hyperplane.
- Continuity: can’t move between budget hyperplanes without introducing bad equilibria.
- Back to ‘price taking’
Needed Assumptions

Assumption

*The mechanism’s outcome function is $C^2$.*

Assumption

$\partial y_i / \partial m_i$ bounded away from zero.

Assumption

*All $m$ are NE for some $\theta$.*

(This requires joint assumptions on the mechanism and $\Theta$.)
Needed Assumptions

Local & global steepness of mech. vs. prefs.
‘Synthetic’ Bounds

\[ y_i(m^*) \pm \epsilon_{ik}(y_{ij}|m^*) \]

Messy, but does the job.
Bounds for the Groves-Ledyard mechanism.
Assumptions

Ready to formalize this theorem...

- A1: (Differentiability) $y_i(m), x_i(m)$ are all twice continuously differentiable.
- A2: (Responsive $y_i$) $\frac{\partial y_i(m)}{\partial m_{ik}}$ is bounded away from zero. (Keeps $\chi_{ik}$ from going vertical.)
- A3: (Rich Domain & Regularity) All $m$ are NE for some $\theta$. 
The Necessary Condition

Theorem

Take any type space $\Theta$ and 1-dimensional mechanism satisfying A1-A3. If the mechanism Nash implements the Walrasian or Lindahl allocations, it must be that

$$x_i(m) \equiv -q_i(m-i)y(m).$$

(Thus, $g_i(m) \equiv 0$.)

Intuition: $q_i$ is a ‘fixed’ price for $i$. Since $y_i$ is bijective in $m_i$, $i$ can pick any $y_i$. Thus, he picks

$$\max_{y_i} u_i(-q_i(m-i)y_i, y_i)$$
One-Dimensional Walrasian Mechanisms

Theorem

Under A1-A3 there do not exist any one-dimensional mechanisms that implement the Walrasian correspondence.

Proof.

- Need $q_1(m-1) \equiv q_2(m-2) \equiv \ldots \equiv q_n(m-n)$
- Only possible if all $q_i$ are constant.
- $p(\Theta)$ is not a singleton; a contradiction.

\[ \square \]

cf. Reichelstein & Reiter & dimensionality results.
One-Dimensional Lindahl Mechanisms

Assumption (A4)
For all $\theta \in \Theta$, $u_i(x_i, y_i|\theta_i) = v_i(y_i|\theta_i) + x_i$
with $v_i' > 0$ and $v_i'' \in (-B, 1/B)$ for some $B > 0$.

Proposition
Under A1-A4 there are no one-dimensional contractive mechanisms that implement the Lindahl correspondence.
Higher-Dimensional Mechanisms

Can we make every message a Nash equilibrium?

- Start with any 1-dim mechanism \((y, x_1, \ldots, x_n)\).
- Add 2nd dimension: \(m_i = (r_i, s_i)\)
- Let \(\tilde{y}(r) = y(r)\)
- Set \(\tilde{x}_i(r, s) = x_i(r) - |s_i|\)
- If \(s_i \neq 0\) it’s not a NE for any \(\theta\)

No ‘linearity theorem’ out of equilibrium \(\Rightarrow\) more freedom
Necessary Conditions: More Dimensions

- Let $M_i = R_i \times S_i$ so that $y : R \to \mathbb{R}$.
- What $(r, s)$ can never be a Nash equilibrium?
- $U_i(r, s) = v_i(y(r)|\theta_i) - q_i(r, s)y(r) - g_i(r, s)$
- Thus, $s^*_i(r, s_{-i})$ solves $\min_{s_i} q_i(s, r) \ast y(r) + g_i(s, r)$.
- Designer can calculate NE of the ‘tax-minimizing game’ $\forall r$.

Note: $(r, s)$ is NOT a NE if:

1. $s$ is not a NE of the tax-minimizing game, or
2. $P_{ik}(r, s) \neq P_{il}(r, s)$ for some $i, k, l$.

Assumption (A3’)

*If m does not satisfy either of the above then m is a NE for some $\theta$.***
Theorem

Under A1, A2, and A3’, for any \((r, s)\) on NE manifold,

\[ x_i(r, s) = -q_i(r, s)y_i(r) - g_i(r, s), \]

where

\[ \frac{dq_i(r, s^*(r))}{dr_i} = 0 \]

and

\[ g_i(r, s) = 0 \]

along the equilibrium manifold.
Stable Mechanism Recipe

Recipe for designing a contractive mechanism:

1. Need bounded concavity \((v_i'' \in (-B, -1/B))\),
2. Start with \(U_i(r, s) := v_i(y(r)) - q_i(r, s)y(r) - g_i(r, s)\),
3. Define best response functions \((\rho_i(r_{-i}, s_{-i}), \sigma_i(r_{-i}, s_{-i}))\).
4. Write down two FOCs:
   \[
   \frac{\partial U_i}{\partial r_i} \equiv \frac{\partial U_i}{\partial s_i} \equiv 0
   \]
5. Differentiate both sides (I.F.T.) and solve system for
   \[
   \left( \frac{\partial \rho_i}{\partial r_j}, \frac{\partial \rho_i}{\partial s_j}, \frac{\partial \sigma_i}{\partial r_j}, \frac{\partial \sigma_i}{\partial s_j} \right)
   \]
Stable Mechanism Recipe

For example:

\[
\frac{\partial \rho_i}{\partial r_j} = \frac{\partial^2 g_i}{\partial s_i^2} \left( -v'_i \frac{\partial y}{\partial r_i} \frac{\partial y}{\partial r_j} + \frac{\partial y}{\partial r_i} \frac{\partial q_i}{\partial r_j} + \frac{\partial^2 g_i}{\partial r_i \partial r_j} \right) - \frac{\partial^2 g_i}{\partial r_i \partial s_i} \frac{\partial^2 g_i}{\partial s_i \partial r_j} \\
\left( \frac{\partial^2 g_i}{\partial r_i \partial s_i} \right)^2 + v''_i \left( \frac{\partial y}{\partial r_i} \right)^2 \frac{\partial^2 g_i}{\partial s_i^2} - \frac{\partial^2 g_i}{\partial r_i^2} \frac{\partial^2 g_i}{\partial s_i^2}
\]

6. Find parameterized functions such that when some parameter gets big,
   a. \( \sum_{j \neq i} \left( \left\| \frac{\partial \rho_j}{\partial r_i} \right\| + \left\| \frac{\partial \sigma_j}{\partial r_i} \right\| \right) < 1 \) and \( \sum_{j \neq i} \left( \left\| \frac{\partial \rho_j}{\partial s_i} \right\| + \left\| \frac{\partial \sigma_j}{\partial s_i} \right\| \right) < 1 \),
   b. \( g_i = 0 \) in equilibrium, and
   c. \( \sum_i q_i = \kappa \) in equilibrium.

7. Give up and hire an RA to do it.
A Contractive Lindahl Mechanism

\[ y(r) = \sum_i r_i \]

\[ q_i(r_{-i}, s_{-i}) = \left( \frac{\kappa}{n} + r_{i-1} - r_{i+1} \right) + \delta \frac{n-1}{n^2} \left( s_{i+1} - \frac{1}{n} r_{i-1} \right) \]

\[ g_i(r, s) = \frac{1}{2} (s_i - \frac{1}{n} r_{i-1})^2 + \frac{\delta}{2} \left( s_{i+1} - \frac{1}{n} r_i \right)^2 \]

**Theorem**

*This implements Lindahl equilibria. If \( \delta \) is sufficiently large it becomes contractive.*

(In fact, this is a ‘stabilized’ Walker mechanism.)
A Contractive Walrasian Mechanism

\[ y_i(r) = (r_{i-1} - r_{i+1}) - \frac{\delta}{n} (s_{i+1} - \frac{n+1}{n} r_i) \]

\[ q_i(s_{-i}) = \frac{1}{n-1} \sum_{j \neq i} s_j \]

\[ g_i(r, s) = (s_i - \delta \frac{n+1}{n^2} \sum_j r_j)^2 \]

**Theorem**

*This implements Walrasian equilibria. As \( \delta \) gets large it becomes contractive.*

The role of \( y_i \) and \( q_i \) ‘reverse’ from Lindahl.
• Stability demands large parameter values. Is this useful?
• Can we make an anonymous contractive mechanism?
• Contractive $\Rightarrow$ unique equilibrium.
  • What if SCC isn’t single-valued?
  • Note: contractiveness depends on $\Theta$.
• Van Essen et al. experiments on “supermodularity”
• Fact remains: supermodularity $\Rightarrow$ stability in the lab
  • Why??
  • Were those mechs. contractive for the chosen prefs?
  • Is there something else about supermodularity?
Final Thoughts

Further reading:

- Reichelstein & Reiter 1988: Some of the same ideas.
- Brock 1980 & G-L 1987: Sufficiency
- Mathevet 2008: Supermodular Mechanism Design
- Van Essen 2009 & Van Essen, Lazzati & Walker 2008

- Ultimate goal: practical mechanism design
- Conversation between experiments & theory.
The End