Designing Stable Mechanisms for Economic Environments

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Decentralization Conference

Motivation

Behavioral   Mechanism Design

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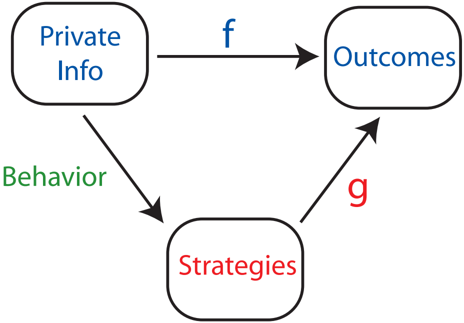


Objective: Design a game so that agents reach some desired objective in equilibrium

Behavioral Mechanism Design

1. Starting point: Groves & Ledyard 1977

1a. Nash implementation

1b. 'Economic' Environments:
Continuity, complexity (message space size), etc.
Differentiability

2. Lesson from experiments: Stability matters
   - Chen & Plott 1996: **'stability' matters**
   - Chen & Tang 1998: **supermodularity**
   - Arifovic & Ledyard 2003: **something weaker**
   - Healy 2006: **dominant diagonal? specific dynamic?**
   - Arifovic & Ledyard 2008: **even weaker...**
   - Current state of knowledge: supermodularity is <u>sufficient</u>.

This Paper

1. Understand how to develop G-L-like mechs.
2. Add 'stability' to the design constraints.

- Economic Environment: Two commodities
  x_i = numéraire, y_i = private or public good
- SCC: Walrasian or Lindahl equilibria (Hurwicz '79)
- Continuously diff'bl mechanisms with 'small' strategy spaces

Theorem 1: Green-Laffont-type necessary cond'n:
$\mathrm{tax}_i(m) = \mathrm{price}_i(m_{-i}) y_i(m)$

Theorem 2: Impossibility results for 1-dimensional m:
WE: No mechanism. LE: No 'stable' mechanism.

Theorem 3: Convert any mechanism into a stable mechanism by adding a dimension to $\mathcal{M}$

The Economic Environment

- Agents: $i \in \{1, 2, \ldots, n\}$.
- Work with net trades; no consumption set boundaries
- Agent i's endowment: $\omega_i = (0, 0)$.
- Net trade vector $z_i = (x_i, y_i)$
    - $x_i \in \mathbb{R}$: numeraire good
    - $y_i \in \mathbb{R}$: non-numeraire good (pub. or pvt)
- Agent i's type: $\theta_i \in \Theta_i$ (complete information.)
- Later: QSL Preferences: $v_i(y_i|\theta_i) + x_i$.
    - v_i is differentiable, strictly concave.

Walrasian & Lindahl Equilibrium

A Walrasian equilibrium is (z^*, p^*) such that
(1) $\sum_i z_i^* = 0$,
(2) each z_i^* maximizes u_i s.t. $x_i + p^* y_i \leq 0$.

Public good: Set $c(y) = \kappa y$.
A Lindahl equilibrium is $(z^*, p_1^*, \ldots, p_n^*)$ such that
(1) $\sum_i x_i^* + \kappa y^* = 0$,
(2) each z_i^* maximizes u_i s.t. $x_i + p_i^* y_i \leq 0$, and
(3) $(\sum_i p_i^*) y - \kappa y$ is maximized at y^*.

Walrasian and Lindahl correspondences: $f : \Theta \twoheadrightarrow \mathcal{Z}$

Mechanisms

- Real-message mechanisms:
  - Strategy space: $\mathcal{M}_i = \mathbb{R}^{K_i} \ \forall i$
  - Outcome functions: $(y_i(m), x_i(m))_i$
- Given a mechanism $(\mathcal{M}, h)$, the Nash correspondence $\nu : \Theta \twoheadrightarrow \mathcal{M}$ identifies the set of Nash equilibria for each θ.
- A mechanism $(\mathcal{M}, h)$ implements a social choice correspondence if $h(\nu(\theta)) = f(\theta)$ for all θ.

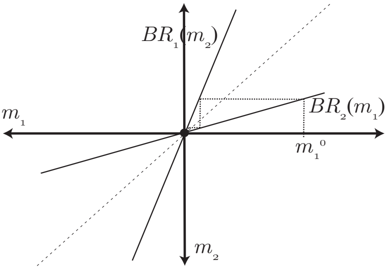
Supermodularity & Stability

Previous literature: supermodularity $\Rightarrow$ stability.
Supermodularity:

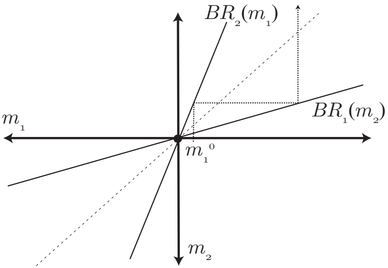
1. $\frac{\partial^2 u_i}{\partial m_{ik} \partial m_{il}} \geq 0$ for all i, $k \neq l$.
2. $\frac{\partial^2 u_i}{\partial m_{ik} \partial m_{jl}} \geq 0$ for all $i \neq j$, k, l.
3. Strategy space is a **closed interval**.

Milgrom & Roberts: 'adaptive dynamics' converge to $[\underline{\mathrm{NE}}, \overline{NE}]$
First 2 conditions: increasing BR curves.
Last condition: ignored in mechanism design!! Problem??

The Power of Supermodularity



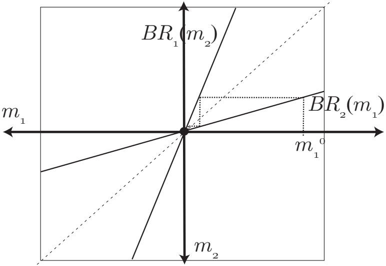
$BR_i(m_j) = \frac{1}{2}m_j$



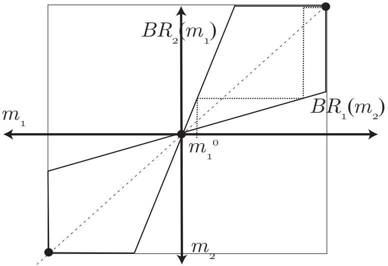
$BR_i(m_j) = 2m_j$

Both games are "supermodular".
Left game is stable, right is not.
Slope of BR curves matters!

The Power of Supermodularity



$BR_i(m_j) = \frac{1}{2}m_j$



$BR_i(m_j) = 2m_j$

Unstable game: boundaries create ‘bad’ (stable) corner equilibria.
‘Stability’ property of supermodularity vacuous here.

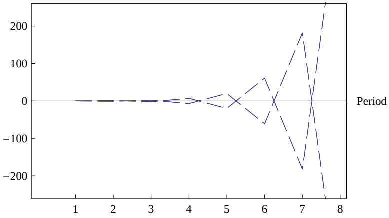
"Counter-Example" Mechanism

Assume $v_i''(\cdot|\theta_i) \in [-M, 0)$ for all $\theta \in \Theta$. Choose

$$y(m) = \sum_{i=1}^{n/2} m_i - \sum_{n/2+1}^{n} m_i$$

$$q_i(m) = \begin{cases} \frac{\kappa}{n} - \gamma \sum_{j \neq \{i, i+\frac{n}{2}\}} m_j & \text{if} \quad i \leq n/2 \\ \frac{\kappa}{n} + \gamma \sum_{j \neq \{i, i+\frac{n}{2}\}} m_j & \text{if} \quad i > n/2. \end{cases}$$

Supermodular if $\gamma > M$. But best response dynamic:



Contractive Mechanisms

Van Essen's suggestion:

- Can we make mechanisms with BR curves that are contraction mappings?
- $||BR(x) - BR(y)|| \leq \alpha ||x - y||$ for $\alpha \in (0, 1)$.
- For now, assume BR is single-valued.

Definition

A mechanism is <u>contractive on Θ</u> if BR is non-empty, closed, and bounded, and for every $\theta \in \Theta$ there exists some $\alpha \in (0, 1)$ such that for every $m, m' \in \mathcal{M}$,

$$||BR(m') - BR(m)|| \leq \alpha ||m' - m||.$$

Does Contractive Imply Stable?

- $\overline{NE} = [\underline{m}^*, \overline{m}^*]$
- Learning dynamic: $\{m(t)\} \subset \mathcal{M}$
- Best response: $BR : \mathcal{M} \to \mathcal{M}$
- $\overline{H}(t', t) = [\min\{m(t'), \ldots, m(t)\}, \max\{m(t'), \ldots, m(t)\}]$
- $\overline{BR}(\mathcal{M}') = [\inf BR(\mathcal{M}'), \sup BR(\mathcal{M}')]$
- $\overline{BR}^*(\mathcal{M}') = \left[\min(\overline{BR}(\mathcal{M}') \cup \overline{NE}), \max(\overline{BR}(\mathcal{M}') \cup \overline{NE})\right]$
- **Adaptive Best Response Dynamic**:
  $(\forall t')(\exists t'')(\forall t \geq t'') \;\; m(t) \in \overline{BR}^*(\overline{H}(t', t-1))$.

Theorem: If $\{m(t)\}$ is an ABR Dynamic and $BR(\cdot)$ is contractive then $m(t)$ converges to $\overline{NE}$.

Back to Mechanisms

OK... how can we make a mechanism contractive?
Step 1: Understand how mechanisms look & feel.

Trivial Observation:

Every mechanism's numeraire outcome functions can be written as

$$x_i(m) = -\underbrace{q_i(m_{-i})}_{\text{`Price'}}\underbrace{y_i(m)}_{\text{`Qty'}} - \underbrace{g_i(m)}_{\text{`Penalty'}} .$$

Note: 'Price-taking' assumption

Groves-Ledyard 1977

$m_i \in \mathbb{R}^1$

$y(m) = \sum_i m_i$

$$q_i(m_{-i}) = \frac{\kappa}{n}$$

$$g_i(m) = \frac{\gamma}{2}\left[\frac{n-1}{n}(m_i - \mu_{-i})^2 - \sigma_{-i}^2\right]$$

$\mu_{-i} = \frac{1}{n-1}\sum_{j\neq i} m_j \qquad \sigma_{-i}^2 = \frac{1}{n-2}\sum_{j\neq i}(m_j - \mu_{-i})^2$

Notes:

- Does not implement Lindahl allocations
- May not be individually rational (see Hurwicz 79)
- Free parameter γ can guarantee stability*

Walker 1981

$m_i \in \mathbb{R}^1$

$y(m) = \sum_i m_i$

$$q_i(m_{-i}) = \frac{\kappa}{n} + m_{i+1} - m_{i+2}$$

$$g_i(m) \equiv 0$$

Notes:

- Implements Lindahl allocations ($\Rightarrow$ IR)
- Uses only one dimension
- Agents are 'price-taking'
- "Wildly" unstable

Hurwicz 1979

$m_i = (s_i, z_i) \in \mathbb{R}^2$

$y_i(m) = s_i - \frac{1}{n-1} \sum_{j \neq i} s_j$

$q_i(m_{-i}) = -\frac{1}{n-1} \sum_{j \neq i} z_j$

$g_i(m) = \left(z_i - \frac{1}{n-1} \sum_{j \neq i} z_j\right)^2 + S_i(m_{-i})$

Notes:

- Implements Walrasian allocations using two dimensions
- $g_i = 0$ in equilibrium
- Agents are 'price-taking'
- No free parameter; may not be stable

Kim 1993

$m_i = (s_i, z_i) \in \mathbb{R}^2$

$y(m) = \sum_i s_i$

$$q_i(m_{-i}) = \frac{\kappa}{n} - \sum_{j \neq i} s_j + \frac{1}{n} \sum_{j \neq i} z_j$$

$$g_i(m) = -\frac{1}{2} \left(z_i - y(m)\right)^2$$

Notes:

- Implements Lindahl using two dimensions
- $g_i = 0$ in equilibrium
- Agents are 'price-taking'
- Globally stable (adjustment process) with QSL prefs

Y. Chen 2002

$m_i = (s_i, z_i) \in \mathbb{R}^2$

$y(m) = \sum_i s_i$

$$q_i(m_{-i}) = \frac{\kappa}{n} - \gamma \sum_{j \neq i} s_j + \frac{\gamma}{n} \sum_{j \neq i} z_j$$

$$g_i(m) = -\frac{1}{2}\left(z_i - y(m)\right)^2 + \frac{\delta}{2} \sum_{j \neq i} (z_j - y(m))^2$$

Notes:

- Implements Lindahl using two dimensions
- $g_i = 0$ in equilibrium
- Agents are 'price-taking'
- Induces supermodular* game with QSL prefs

Others...

Other mechanisms:

- de Trenqualye 1989
- Vega-Redondo 1989
- Kim 1996
- Corchon & Wilkie 1996

In all of these...

- Implement Lindahl using two dimensions
- $g_i = 0$ in equilibrium
- Agents are ‘price-taking’

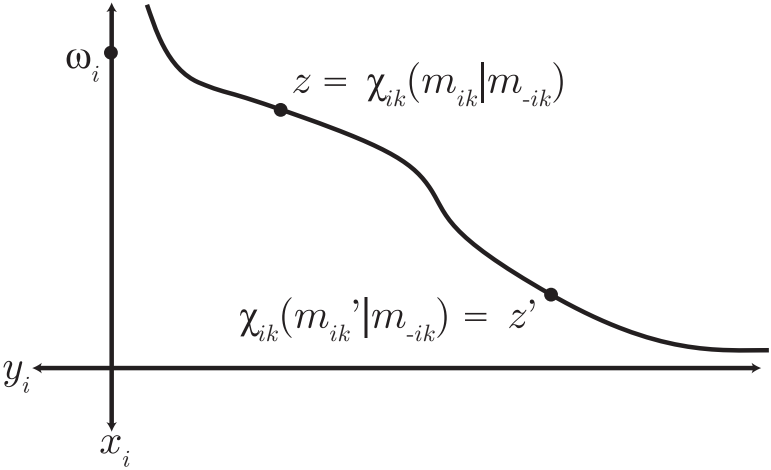
Eerie Similarities

Why are these mechanisms so similar?

How do they work?

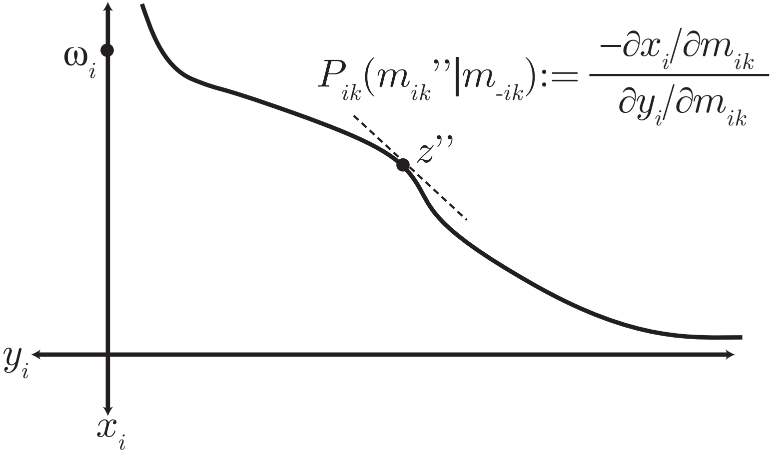
How much freedom is there to play with them?

The Graphical View



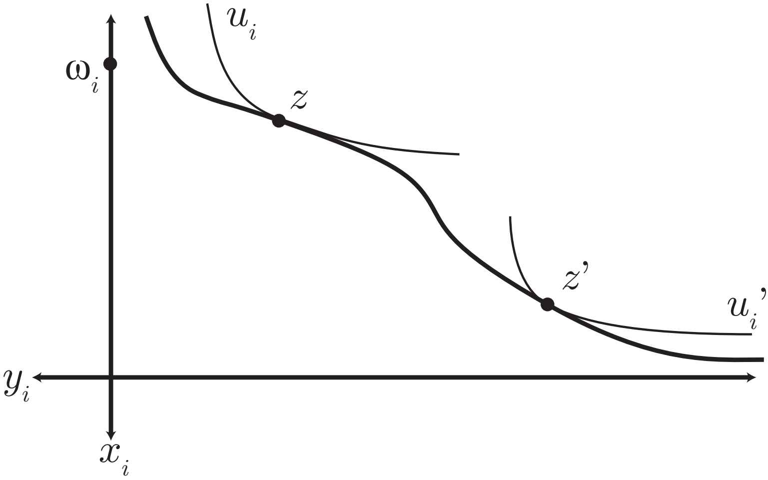
What you can achieve by changing m_i (given m_{-i})

The Local Price



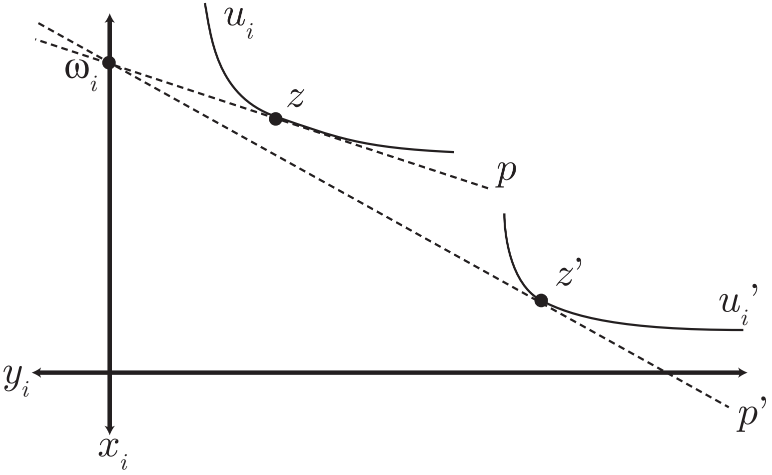
Slope of χ_i is the 'local price'.

Nash Equilibrium Points



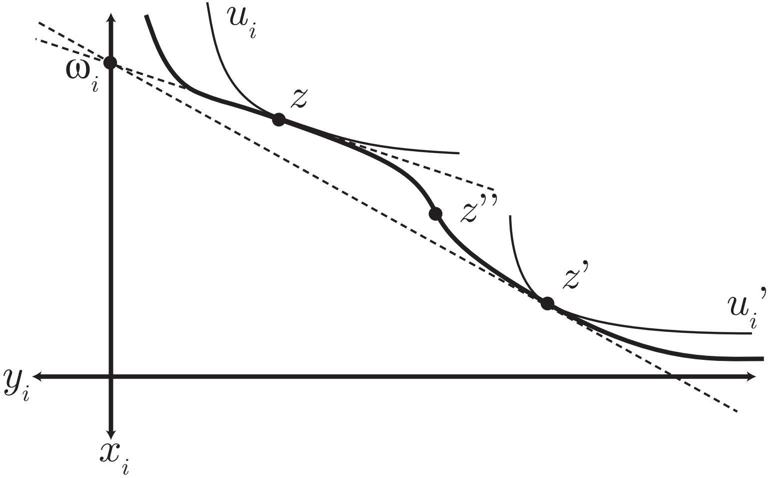
Possible Nash equilibrium points given u_i or u_i'.

Walrasian Allocations



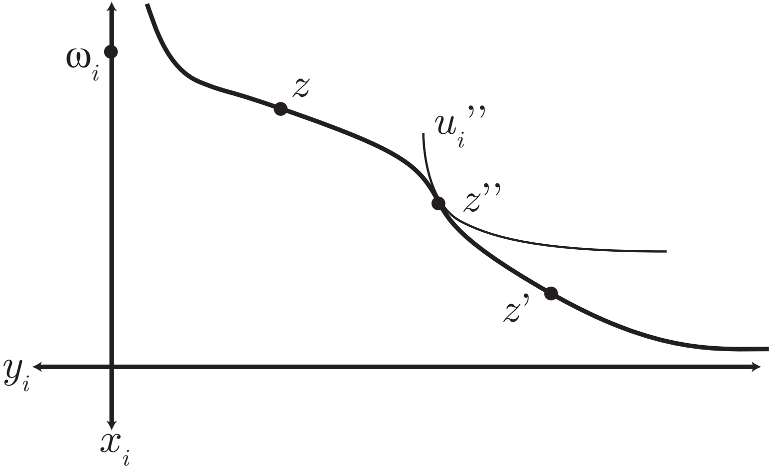
Possible Walrasian allocations given u_i or u_i'.

Nash Implementation



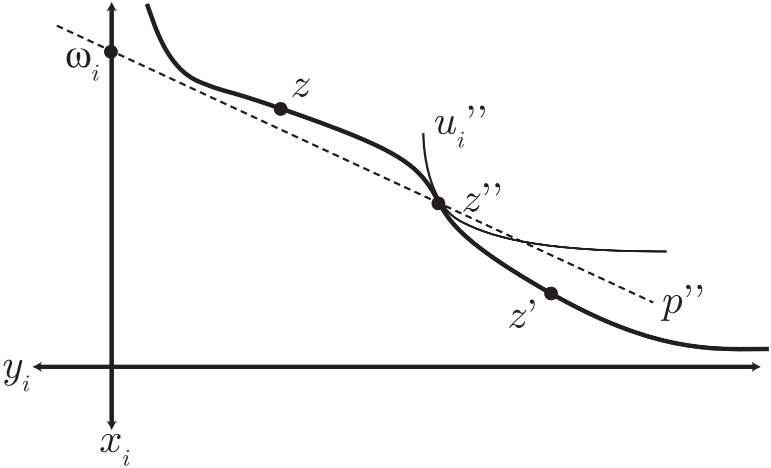
Triple tangency is necessary for NE outcome to be WE.

'Bad' Nash Equilibria



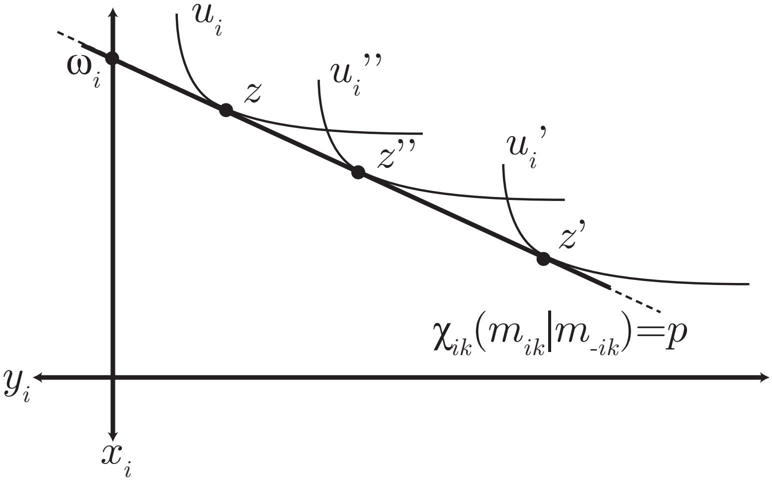
Rich-enough type space $\Rightarrow$ ANY *m* is a NE.

'Bad' Nash Equilibria



But now the mechanism doesn't implement Walrasian allocations!

The Necessary Condition



Only way to avoid 'bad' equilibria: $t_i(m) = q_i(m_{-i})y_i(m)$.

Conclusion So Far

- If every strategy is an equilibrium, all equilibrium outcomes need to be on a budget hyperplane.
- Continuity: can't move between budget hyperplanes without introducing bad equilibria.
- Back to 'price taking'

Needed Assumptions

Assumption

The mechanism's outcome function is $\mathcal{C}^2$*.*

Assumption

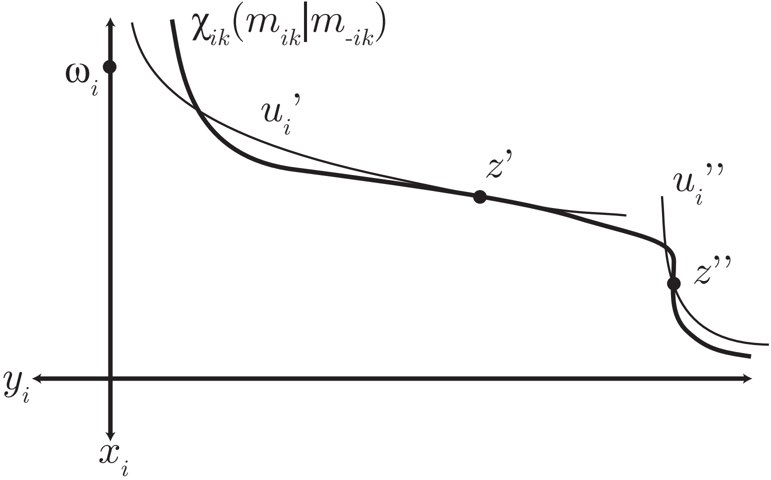
$\partial y_i / \partial m_i$ *bounded away from zero.*

Assumption

All m *are NE for some* θ*.*

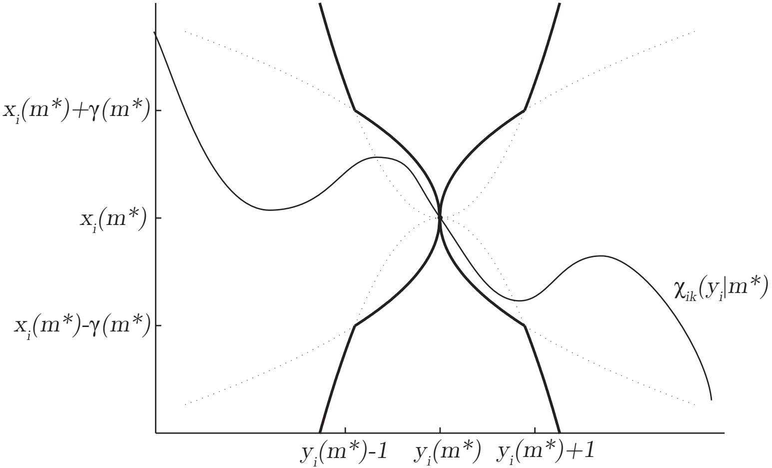
(This requires joint assumptions on the mechanism <u>and</u> Θ.)

Needed Assumptions



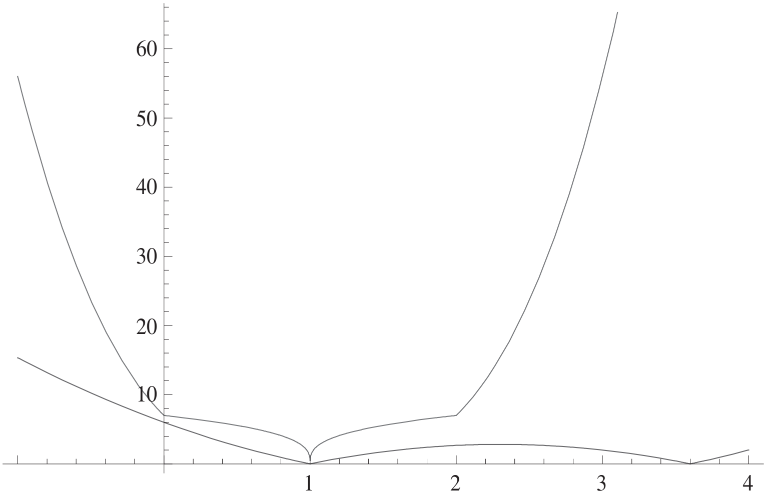
Local & global steepness of mech. vs. prefs.

'Synthetic' Bounds



Messy, but does the job.

The Bounds At Work



Bounds for the Groves-Ledyard mechanism.

Assumptions

Ready to formalize this theorem...

- A1: (Differentiability) $y_i(m)$, $x_i(m)$ are all twice continuously differentiable.
- A2: (Responsive y_i) $\partial y_i(m)/\partial m_{ik}$ is bounded away from zero. (Keeps χ_{ik} from going vertical.)
- A3: (Rich Domain & Regularity) All m are NE for some θ.

The Necessary Condition

Theorem

Take any type space Θ and 1-dimensional mechanism satisfying A1-A3. If the mechanism Nash implements the Walrasian or Lindahl allocations, it must be that

$$x_i(m) \equiv -q_i(m_{-i})y(m).$$

(Thus, $g_i(m) \equiv 0$.)

Intuition: q_i is a 'fixed' price for i. Since y_i is bijective in m_i, i can pick any y_i. Thus, he picks

$$\max_{y_i} u_i(-q_i(m_{-i})y_i, y_i)$$

One-Dimensional Walrasian Mechanisms

Theorem

Under A1-A3 there do not exist any one-dimensional mechanisms that implement the Walrasian correspondence.

Proof.

- Need $q_1(m_{-1}) \equiv q_2(m_{-2}) \equiv \ldots \equiv q_n(m_{-n})$
- Only possible if all q_i are constant.
- $p(\Theta)$ is not a singleton; a contradiction.

$\square$

cf. Reichelstein & Reiter & dimensionality results.

One-Dimensional Lindahl Mechanisms

Assumption (A4)

For all $\theta \in \Theta$, $u_i(x_i, y_i|\theta_i) = v_i(y_i|\theta_i) + x_i$
with $v_i' > 0$ *and* $v_i'' \in (-B, 1/B)$ *for some* $B > 0$.

Proposition

Under A1-A4 there are no one-dimensional contractive mechanisms that implement the Lindahl correspondence.

Higher-Dimensional Mechanisms

Can we make every message a Nash equilibrium?

- Start with any 1-dim mechanism $(y, x_1, \ldots, x_n)$.
- Add 2nd dimension: $m_i = (r_i, s_i)$
- Let $\tilde{y}(r) = y(r)$
- Set $\tilde{x}_i(r, s) = x_i(r) - |s_i|$
- If $s_i \neq 0$ it's not a NE for <u>any</u> θ

No 'linearity theorem' out of equilibrium $\Rightarrow$ more freedom

Necessary Conditions: More Dimensions

- Let $\mathcal{M}_i = \mathcal{R}_i \times \mathcal{S}_i$ so that $y : \mathcal{R} \to \mathbb{R}$.
- What (r, s) can never be a Nash equilibrium?
- $U_i(r, s) = v_i(y(r)|\theta_i) - q_i(r, s)y(r) - g_i(r, s)$
- Thus, $s_i^*(r, s_{-i})$ solves $\min_{s_i} q_i(s, r) * y(r) + g_i(s, r)$.
- Designer can calculate NE of the 'tax-minimizing game' $\forall r$.

Note: (r, s) is NOT a NE if:

1. s is not a NE of the tax-minimizing game, or
2. $P_{ik}(r, s) \neq P_{il}(r, s)$ for some i, k, l.

Assumption (A3')

If m does not satisfy either of the above then m is a NE for some θ.

More Dimensions

Theorem

Under A1, A2, and A3', for any (r, s) on NE manifold,

$$x_i(r,s) = -q_i(r,s)y_i(r) - g_i(r,s),$$

where

$$\frac{dq_i(r,s^*(r))}{dr_i} = 0$$

and

$$g_i(r,s) = 0$$

along the equilibrium manifold.

Stable Mechanism Recipe

Recipe for designing a contractive mechanism:

1. Need bounded concavity ($v_i'' \in (-B, -1/B)$),
2. Start with $U_i(r,s) := v_i(y(r)) - q_i(r,s)y(r) - g_i(r,s)$
3. Define best response functions $(\rho_i(r_{-i}, s_{-i}), \sigma_i(r_{-i}, s_{-i}))$.
4. Write down two FOCs:

$$\frac{\partial U_i(\rho_i, \sigma_i, r_{-i}, s_{-i})}{\partial r_i} \equiv \frac{\partial U_i(\rho_i, \sigma_i, r_{-i}, s_{-i})}{\partial s_i} \equiv 0$$

5. Differentiate both sides (I.F.T.) and solve system for

$$\left(\frac{\partial \rho_i}{\partial r_j}, \frac{\partial \rho_i}{\partial s_j}, \frac{\partial \sigma_i}{\partial r_j}, \frac{\partial \sigma_i}{\partial s_j} \right)$$

Stable Mechanism Recipe

For example:

$$\frac{\partial \rho_i}{\partial r_j} = \frac{\frac{\partial^2 g_i}{\partial s_i^2}\left(-v_i'' \frac{\partial y}{\partial r_i}\frac{\partial y}{\partial r_j} + \frac{\partial y}{\partial r_i}\frac{\partial q_i}{\partial r_j} + \frac{\partial^2 g_i}{\partial r_i \partial r_j}\right) - \frac{\partial^2 g_i}{\partial r_i \partial s_i}\frac{\partial^2 g_i}{\partial s_i \partial r_j}}{\left(\frac{\partial^2 g_i}{\partial r_i \partial s_i}\right)^2 + v_i''\left(\frac{\partial y}{\partial r_i}\right)^2 \frac{\partial^2 g_i}{\partial s_i^2} - \frac{\partial^2 g_i}{\partial r_i^2}\frac{\partial^2 g_i}{\partial s_i^2}}$$

6. Find parameterized functions such that when some parameter gets big,
   a. $\sum_{j \neq i}\left(\left|\frac{\partial \rho_j}{\partial r_i}\right| + \left|\frac{\partial \sigma_j}{\partial r_i}\right|\right) < 1$ and $\sum_{j \neq i}\left(\left|\frac{\partial \rho_j}{\partial s_i}\right| + \left|\frac{\partial \sigma_j}{\partial s_i}\right|\right) < 1$,
   b. $g_i = 0$ in equilibrium, and
   c. $\sum_i q_i = \kappa$ in equilibrium.
7. Give up and hire an RA to do it.

A Contractive Lindahl Mechanism

$$
\begin{aligned}
y(r) &= \textstyle\sum_i r_i \\
q_i(r_{-i}, s_{-i}) &= \left(\frac{\kappa}{n} + r_{i-1} - r_{i+1}\right) + \delta \frac{n-1}{n^2}\left(s_{i+1} - \frac{1}{n} r_{i-1}\right) \\
g_i(r, s) &= \frac{1}{2}\left(s_i - \frac{1}{n} r_{i-1}\right)^2 + \frac{\delta}{2}\left(s_{i+1} - \frac{1}{n} r_i\right)^2
\end{aligned}
$$

Theorem

This implements Lindahl equilibria. If δ is sufficiently large it becomes contractive.

(In fact, this is a ‘stabilized’ Walker mechanism.)

A Contractive Walrasian Mechanism

$$
\begin{aligned}
y_i(r) &= (r_{i-1} - r_{i+1}) - \tfrac{\delta}{n}\left(s_{i+1} - \tfrac{n+1}{n} r_i\right) \\
q_i(s_{-i}) &= \tfrac{1}{n-1} \textstyle\sum_{j \neq i} s_j \\
g_i(r, s) &= (s_i - \delta \tfrac{n+1}{n^2} \textstyle\sum_j r_j)^2
\end{aligned}
$$

Theorem

This implements Walrasian equilibria. As δ gets large it becomes contractive.

The role of y_i and q_i 'reverse' from Lindahl.

Notes on this Procedure

- Stability demands large parameter values. Is this useful?
- Can we make an anonymous contractive mechanism?
- Contractive $\Rightarrow$ unique equilibrium.
  - What if SCC isn't single-valued?
  - Note: contractiveness depends on Θ.
- Van Essen et al. experiments on "supermodularity"
- Fact remains: supermodularity $\Rightarrow$ stability in the lab
  - Why??
  - Were those mechs. contractive for the chosen prefs?
  - Is there something else about supermodularity?

Final Thoughts

Further reading:

- Reichelstein & Reiter 1988: Some of the same ideas.
- Brock 1980 & G-L 1987: Sufficiency
- Mathevet 2008: Supermodular Mechanism Design
- Van Essen 2009 & Van Essen, Lazzati & Walker 2008

- Ultimate goal: practical mechanism design
- Conversation between experiments & theory.

The End