Updating Toward the Signal

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Midwest Economic Theory
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The Setting

- X = some random variable of interest
- $Z = X + \tilde{\varepsilon}$ = noisy signal of X
- $E[\tilde{\varepsilon}|X = x] = 0 \ \forall x$
- Care about $E[X|Z = z]$
- Often assumed that $E[X|z] = \alpha z + (1 - \alpha)E[X]$
- When is this appropriate? Is it robust?

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  - X = avg score of the population a quiz
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  - z large $\Rightarrow E[X] < E[X|z] < z$ (overconfidence)
- Shapiro (1986) or Novshek & Sonnenschein (1982)
  - Oligopolist observes noisy signal of cost or demand
  - Good signal $\Rightarrow$ competitive advantage
- Morris & Shin (2000, 2002, 2006)
  - Global Games: additive noise in information about state
  - Value of public info: welfare effects of signal precision
  - Uses normal distributions, guaranteeing $E[X|z] = \alpha z + (1-\alpha)E[X]$

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Assumptions

- Assume all r.v.'s are real-valued and have cts densities & finite means
- Consider *families* of error terms $\mathcal{E}$
- **Questions:** What conditions on X and $\mathcal{E}$ guarantee

$$E[X|z] = \alpha z + (1-\alpha)E[X] \text{ ?} \tag{1}$$

  Does (1) imply anything about X or $\mathcal{E}$?
- Relevant properties of r.v.'s:
  - *Symmetric:* symmetric density about the mean
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Definitions

Definition

X **updates toward the signal w.r.t** $\mathcal{E}$ (**UTS-**$\mathcal{E}$) if $\forall \tilde{\varepsilon} \in \mathcal{E}$, $\forall z$ $\exists \alpha \in [0, 1]$ s.t.

$$E[X|Z = z] = \alpha z + (1 - \alpha)E[X]. \tag{2}$$

Definition

X **updates in the direction of the signal w.r.t** $\mathcal{E}$ (**UDS-**$\mathcal{E}$) if equation (2) holds with $\alpha \geq 0$ $\forall \tilde{\varepsilon} \in \mathcal{E}$.

Definition

X satisfies **mean reinforcement with respect to** $\mathcal{E}$ (**MR-**$\mathcal{E}$) if $\forall \tilde{\varepsilon} \in \mathcal{E}$

$$E[X|z = E[X]] = E[X] \tag{3}$$

Families of Error Terms

All error terms are continuous, mean-zero, and satisfy sym. dep.:

Definition

$\tilde{\varepsilon}$ satisfies **symmetric dependence** if, for almost every $\varepsilon, a \in \mathbb{R}$, $f_{\tilde{\varepsilon}}(\varepsilon | z = E[X] + a) = f_{\tilde{\varepsilon}}(\varepsilon | z = E[X] - a)$.

- $\mathcal{E}_S$ = all symmetric error terms.
- $\mathcal{E}_{S,Q}$ = all symmetric, quasiconcave error terms.
- $\mathcal{E}_{S,Q,I}$ = all symmetric, quasiconcave error terms indep. of X.
- $\mathcal{E}_{2pt}$ = all two-point distributions of the form $(-y, p; y, 1-p)$.
- $\mathcal{E}_U$ = all uniform distributions of the form $U[-y, y]$.

For any $\mathcal{E}$, $\overline{\mathcal{E}}$ = the weak closure of $\mathcal{E}$.

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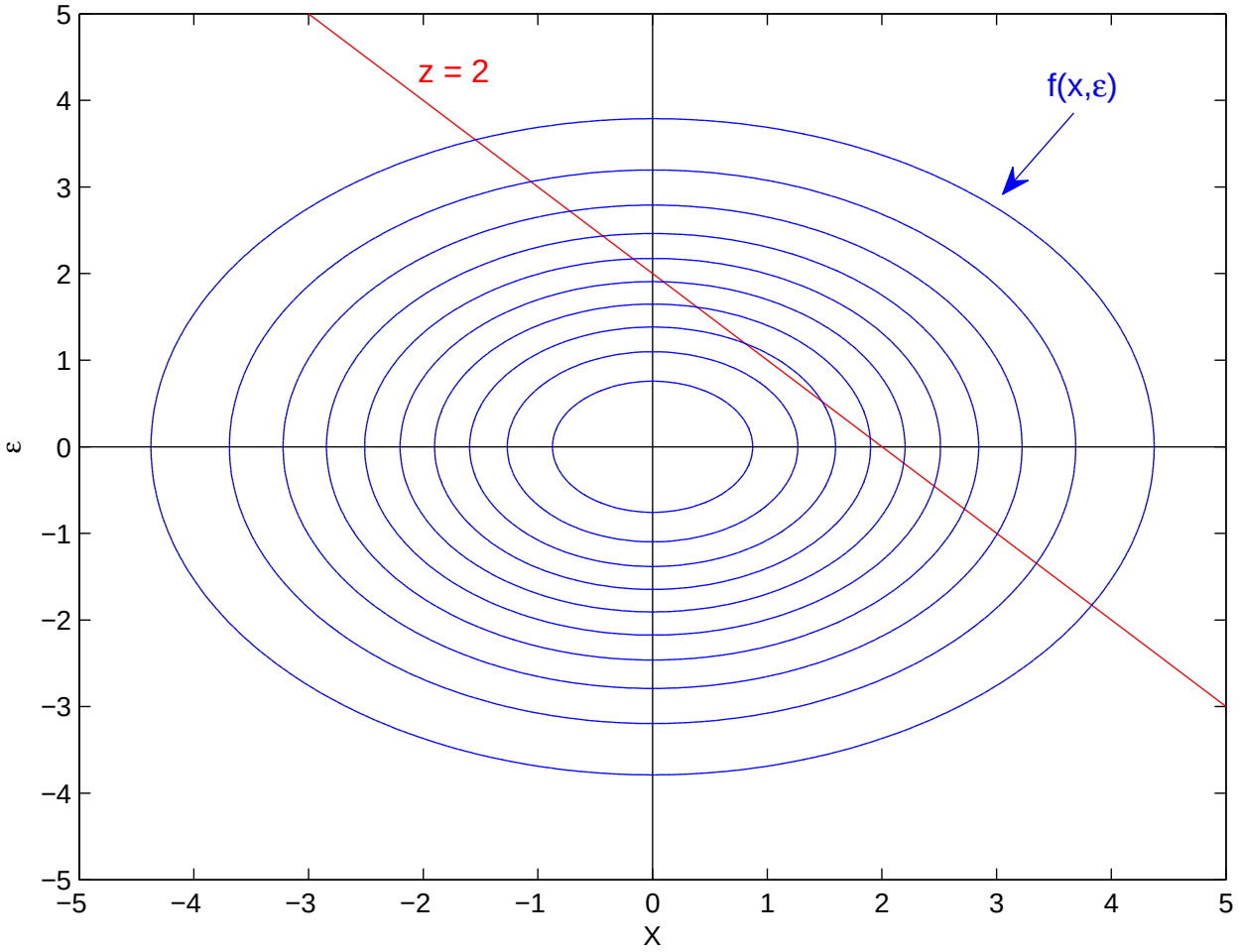
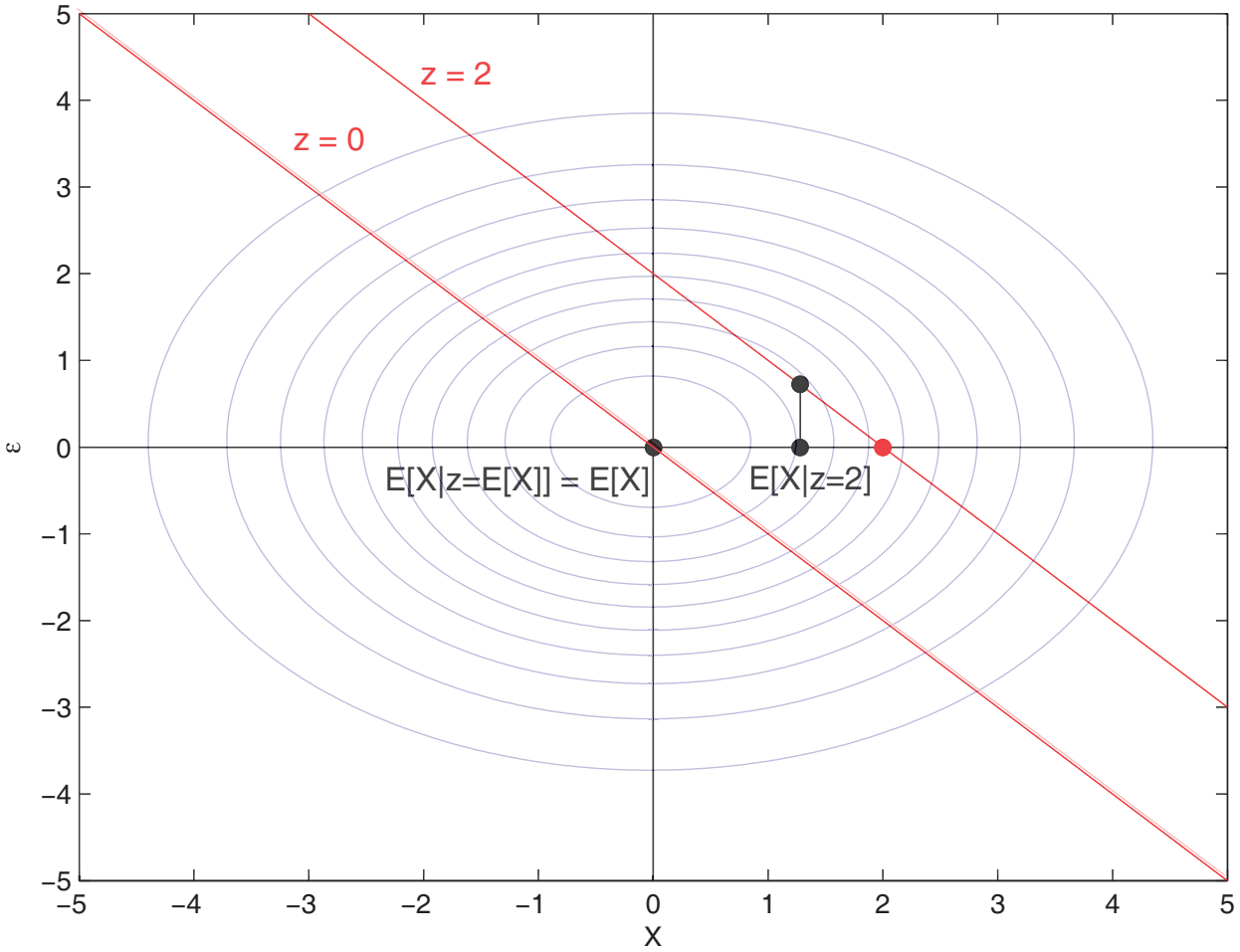
Visualizing the Conditions



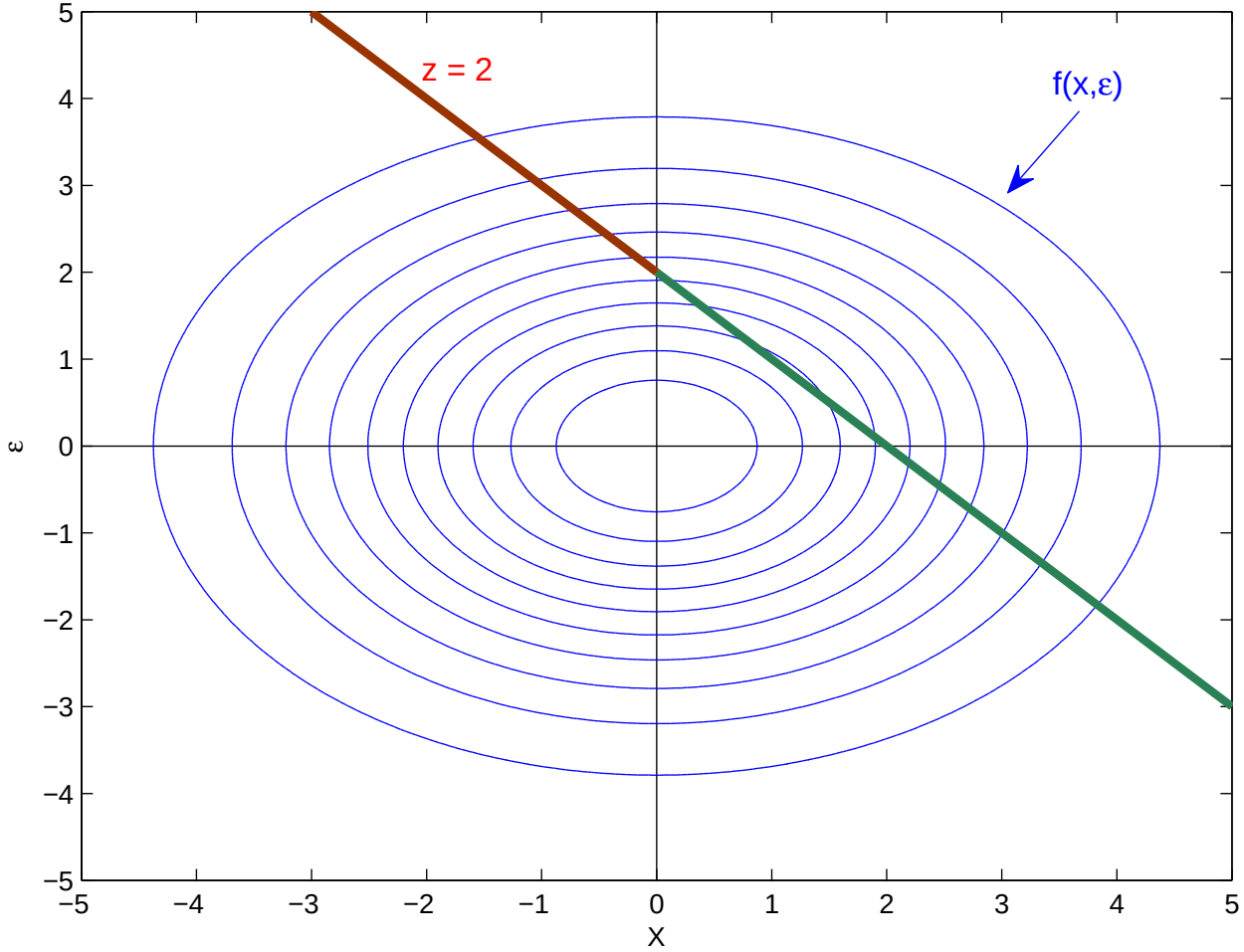
$$f(x|z) = \frac{f_{\tilde{\varepsilon}}(z-x|x)f_X(x)}{\int f_{\tilde{\varepsilon}}(z-\xi|\xi)f_X(\xi)d\xi} \text{ so } E[X|z] = \int \frac{x\ f_{\tilde{\varepsilon}}(z-x|x)f_X(x)}{\int f_{\tilde{\varepsilon}}(z-\xi|\xi)f_X(\xi)d\xi}dx$$

The Normal-Normal Case



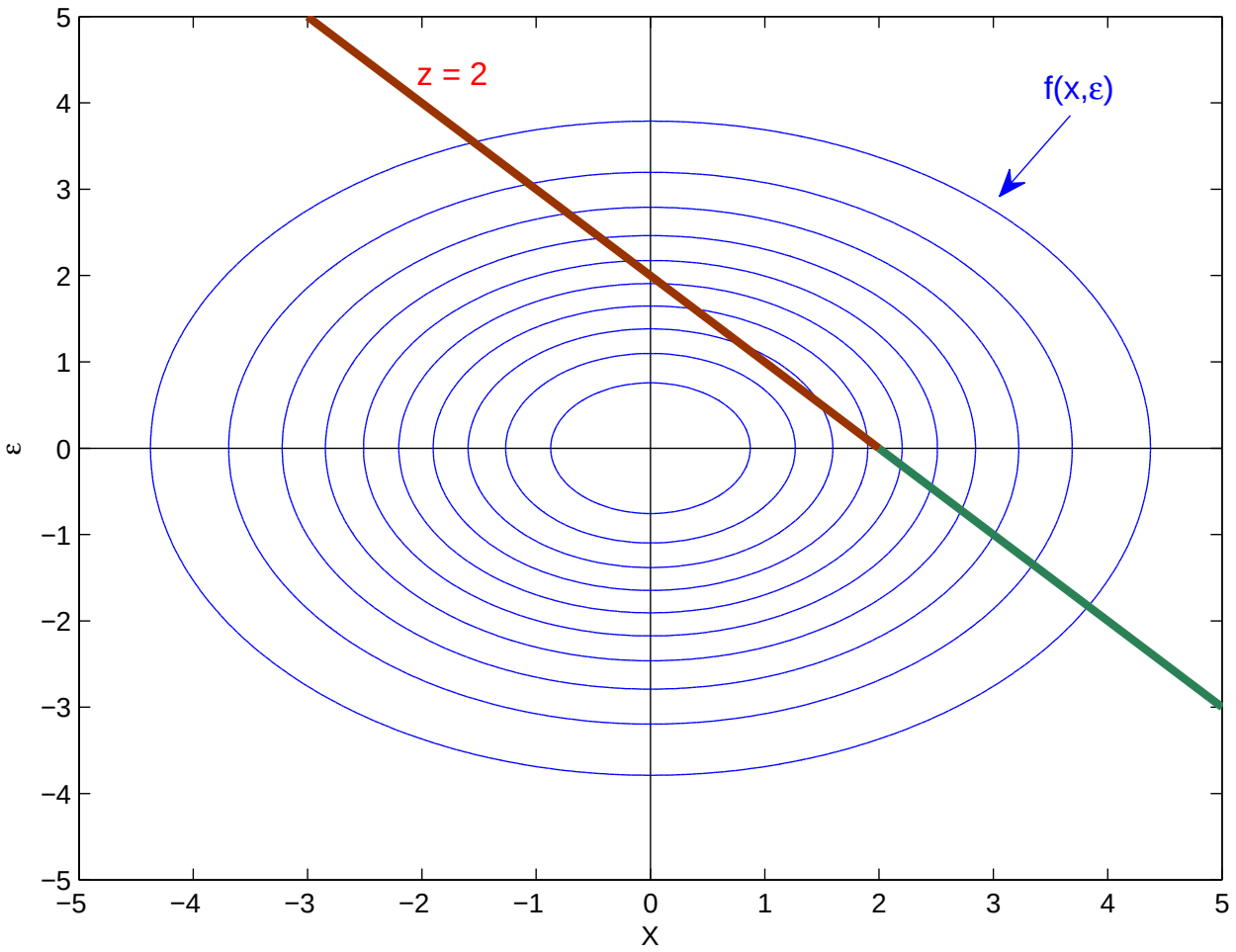
If $X \sim \mathcal{N}(0,2)$ and $\tilde{\varepsilon} \sim \mathcal{N}(0,1)$ then $E[X|z=2] = 1.6$

Visualizing the Conditions



$$\int_{-\infty}^{0} -x \; f_{\tilde{\varepsilon}}(z-x|x) f_X(x) dx \leq \int_{0}^{\infty} x \; f_{\tilde{\varepsilon}}(z-x|x) f_X(x) dx$$

Visualizing the Conditions



Integrate on ε: $E[X|z] \leq z \Leftrightarrow E[\tilde{\varepsilon}|z] \geq 0$

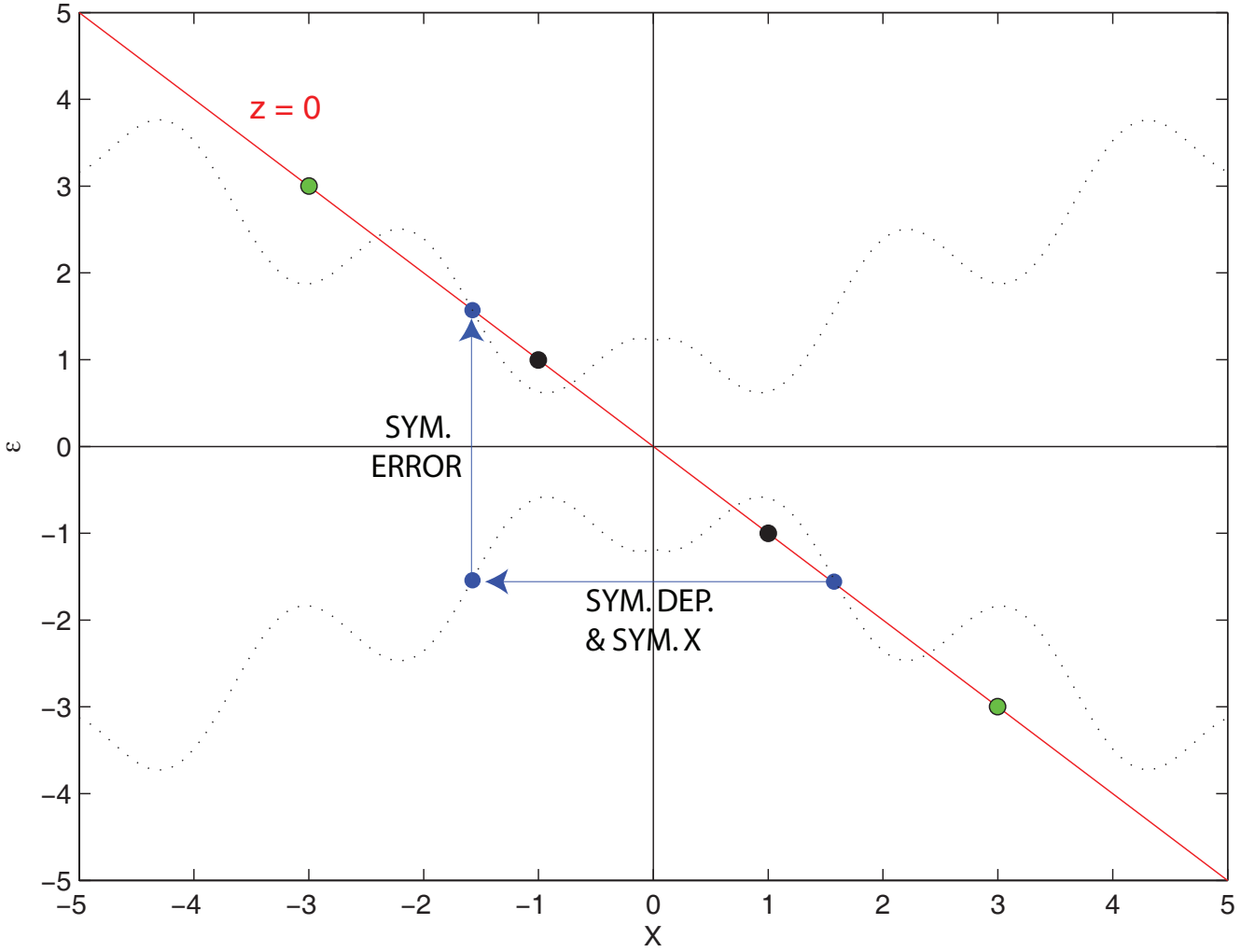
MR: Sufficient Conditions

Proposition

If X is symmetric then X satisfies MR-$\mathcal{E}_S$

Proof.

See pictures... $\square$



$$\int_{-\infty}^{0} -x \; f_{\tilde{\varepsilon}}(0-x|x) f_X(x) dx = \int_{0}^{\infty} x \; f_{\tilde{\varepsilon}}(0-x|x) f_X(x) dx$$

UDS: Sufficient Conditions

X symmetric $\not\Rightarrow$ X UDS-$\mathcal{E}_S$.

Example

Let $f_X(x) = \begin{cases} \frac{1}{3}\left(1 - \frac{|x|}{3}\right) & \text{if} \quad x \in [-3, 3] \\ 0 & \text{otherwise} \end{cases}$

and $\tilde{\varepsilon} = (-2, \frac{1}{2}; 2, \frac{1}{2})$. Then $E[X|z] = -z$, so UDS fails.

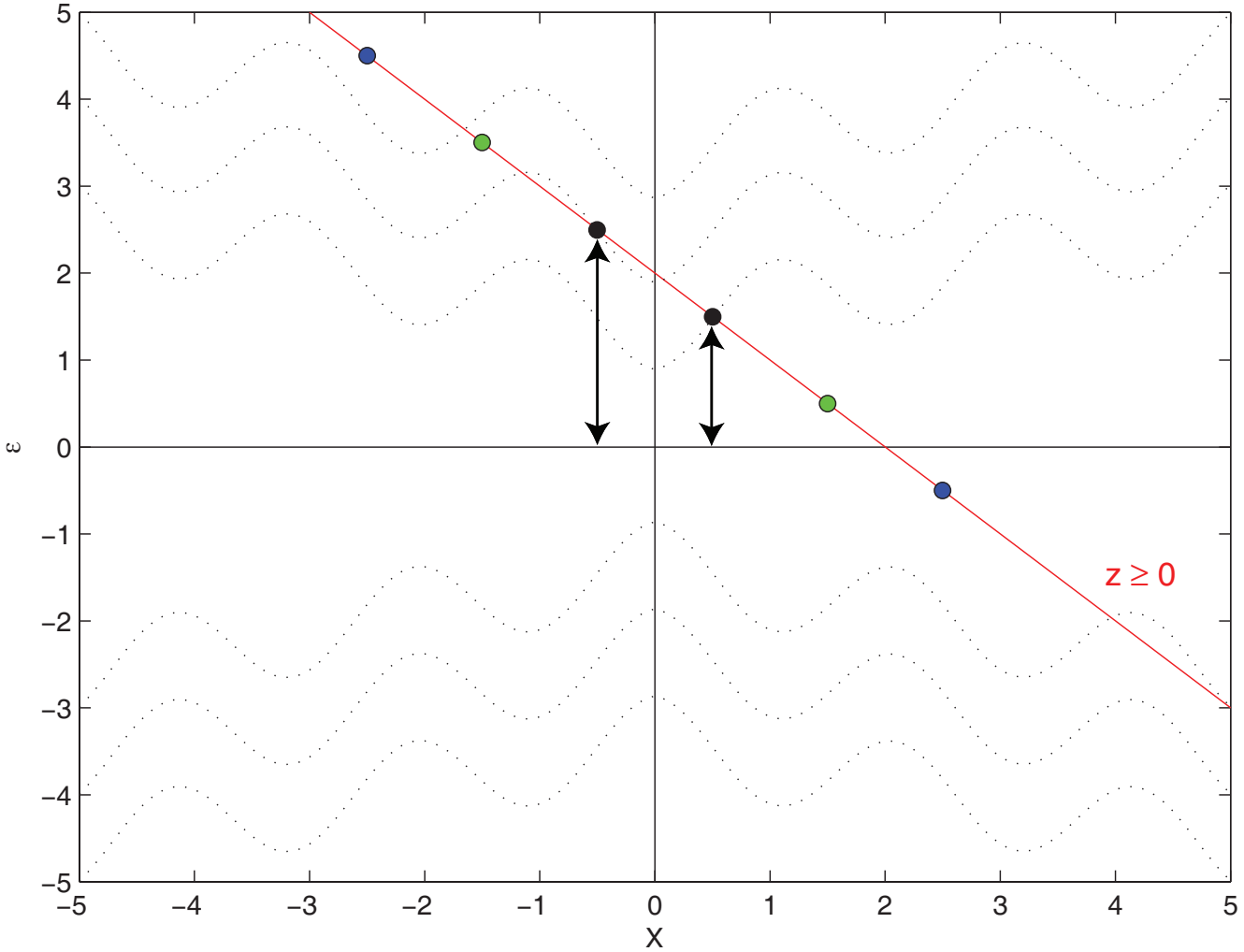
To get UDS, need another restriction on errors:

Proposition

If X is symmetric then X satisfies UDS-$\mathcal{E}_{S,Q}$

Proof (Sketch).

See picture... $\square$



$$z \geq 0 \Rightarrow \int_0^{\infty} xf(x|z)dx \geq -\int_{-\infty}^{0} xf(x|z)dx$$

UTS: Sufficient Conditions

X symmetric $\not\Rightarrow$ X UTS-$\mathcal{E}_{S,Q}$.

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and $X = (-2, 1/2; 2, 1/2)$. Then $E[X|z] = 2z$, so UTS fails.

Proposition

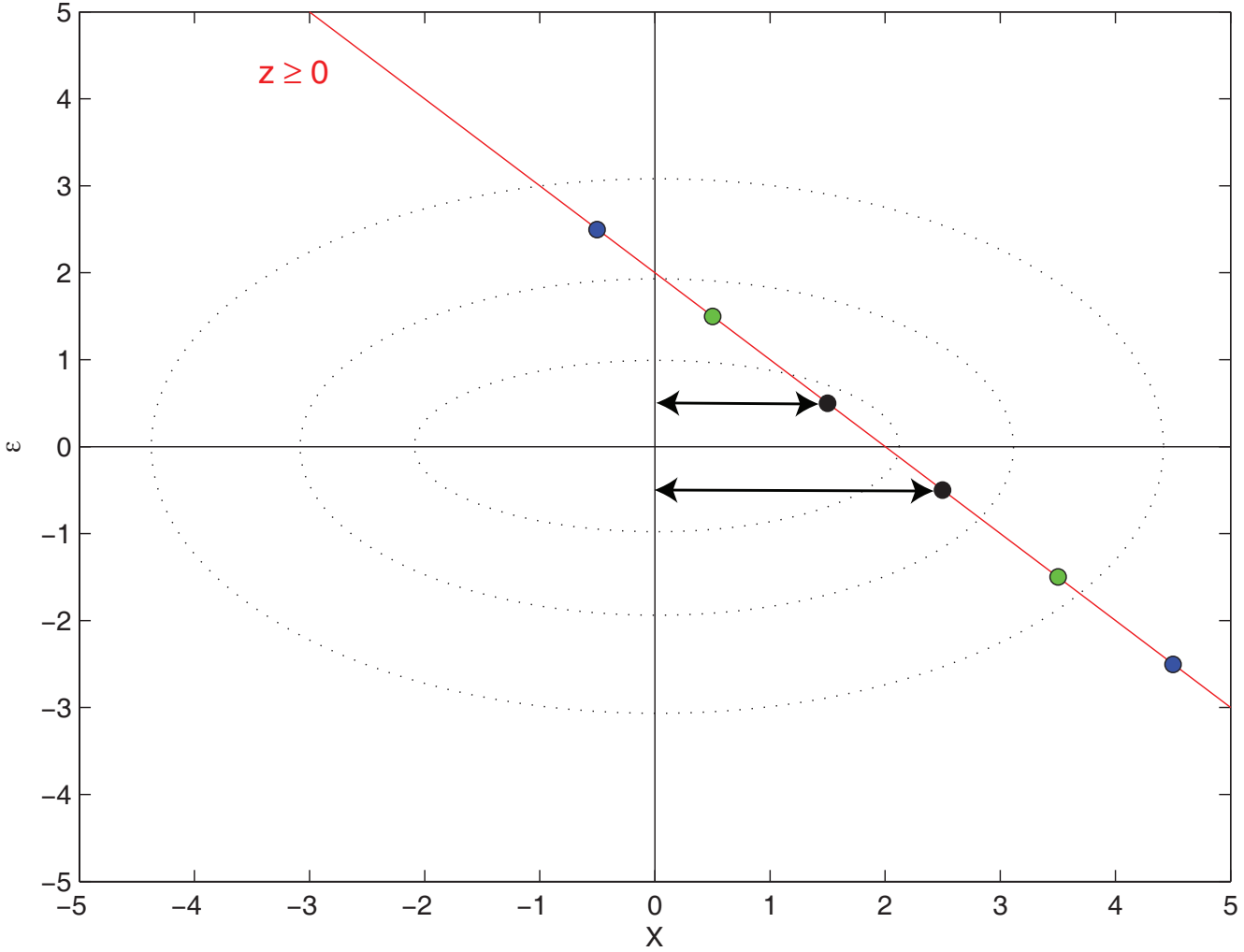
If X is symmetric and quasiconcave then X satisfies UTS-$\mathcal{E}_{S,Q,I}$.

Proof (Sketch).

Already have UDS.

Need to show $E[X|z] \leq z$ when $z \geq 0$.

$\square$



Does ∃ a sufficient condition weaker than independence??

Summary of Results

Family of Error Terms			Prior			Condition
Sym			Sym		$\Rightarrow$	MR
Sym			Sym		$\not\Rightarrow$	UDS
Sym	QC		Sym		$\Rightarrow$	UDS
Sym	QC		Sym		$\not\Rightarrow$	UTS
Sym	QC	Ind*	Sym	QC	$\Rightarrow$	UTS

MR: Necessary Conditions 1

Proposition

Pick any $\mathcal{E}$ with $\mathcal{E}_{2pt} \subseteq \overline{\mathcal{E}}$.
If X satisfies MR-$\mathcal{E}$ then X is symmetric.

Proof.

Pick any $y > 0$ and let $\tilde{\varepsilon} \sim (-y, \frac{1}{2}; y, \frac{1}{2})$.

$z = 0$ means $x \in \{-y, y\}$.

Thus, $E[X|z = 0] \propto -yf_X(-y) + yf_X(y)$.

MR-$\mathcal{E}$ means $-y\, f_X(-y) + y\, f_X(y) = 0$ for every $y > 0$.

Thus, $f_X(y) = f_X(-y)$, so X is symmetric.

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UDS: An Impossibility Result

Proposition

If $\mathcal{E}_{2pt} \subseteq \overline{\mathcal{E}}$ then there does not exist an X such that X UDS-$\mathcal{E}$.

Proof.

Pick any $x_2 > x_1 > 0$ and let $\tilde{\varepsilon} \sim (-\frac{x_1+x_2}{2}, \frac{1}{2}; \frac{x_1+x_2}{2}, \frac{1}{2})$

If $z = (x_2 - x_1)/2$ then $E[X|z] \propto -x_1 f_X(-x_1) + x_2 f_X(x_2)$

By symmetry (prev. proposition), this is $-x_1 f_X(x_1) + x_2 f_X(x_2)$.

UDS $\Rightarrow \geq 0$, so $x_2 f_X(x_2) \geq x_1 f(x_1)$

But then $x f_X(x)$ is increasing, so $E[X] = \int x f_X(x) dx$ does not exist

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MR: Necessary Conditions 2

Proposition

Pick any $\mathcal{E}$ with $\mathcal{E}_U \subseteq \overline{\mathcal{E}}$.
If X satisfies MR-$\mathcal{E}$ then X is symmetric.

Proof.

Pick any $y > 0$ and let $\tilde{\varepsilon} \sim U[-y, y]$.

$z = 0$ means $x \in [-y, y]$.

Thus, $E[X|z = 0] \propto \int_{-y}^{y} x\, f_X(x)\, dx$.

MR-$\mathcal{E}$ means $\int_{-y}^{y} x\, f_X(x)\, dx = 0$ for every y.

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If $\mathcal{E} \subseteq \mathcal{E}_{S,Q}$ then MR-$\mathcal{E} \Rightarrow$ Sym Prior $\Rightarrow$ UDS-$\mathcal{E} \Rightarrow$ MR-$\mathcal{E}$.

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UTS: Necessary Conditions

Proposition

Pick any $\mathcal{E}$ with $\mathcal{E}_U \subseteq \overline{\mathcal{E}}$.
If X satisfies UTS-$\mathcal{E}$ then X is symmetric **and quasiconcave**.

Proof (Sketch).

- UTS-$\mathcal{E}$ $\Rightarrow$ MR-$\mathcal{E}$ $\Rightarrow$ X symmetric. ✓
- For quasiconcavity, see pictures...

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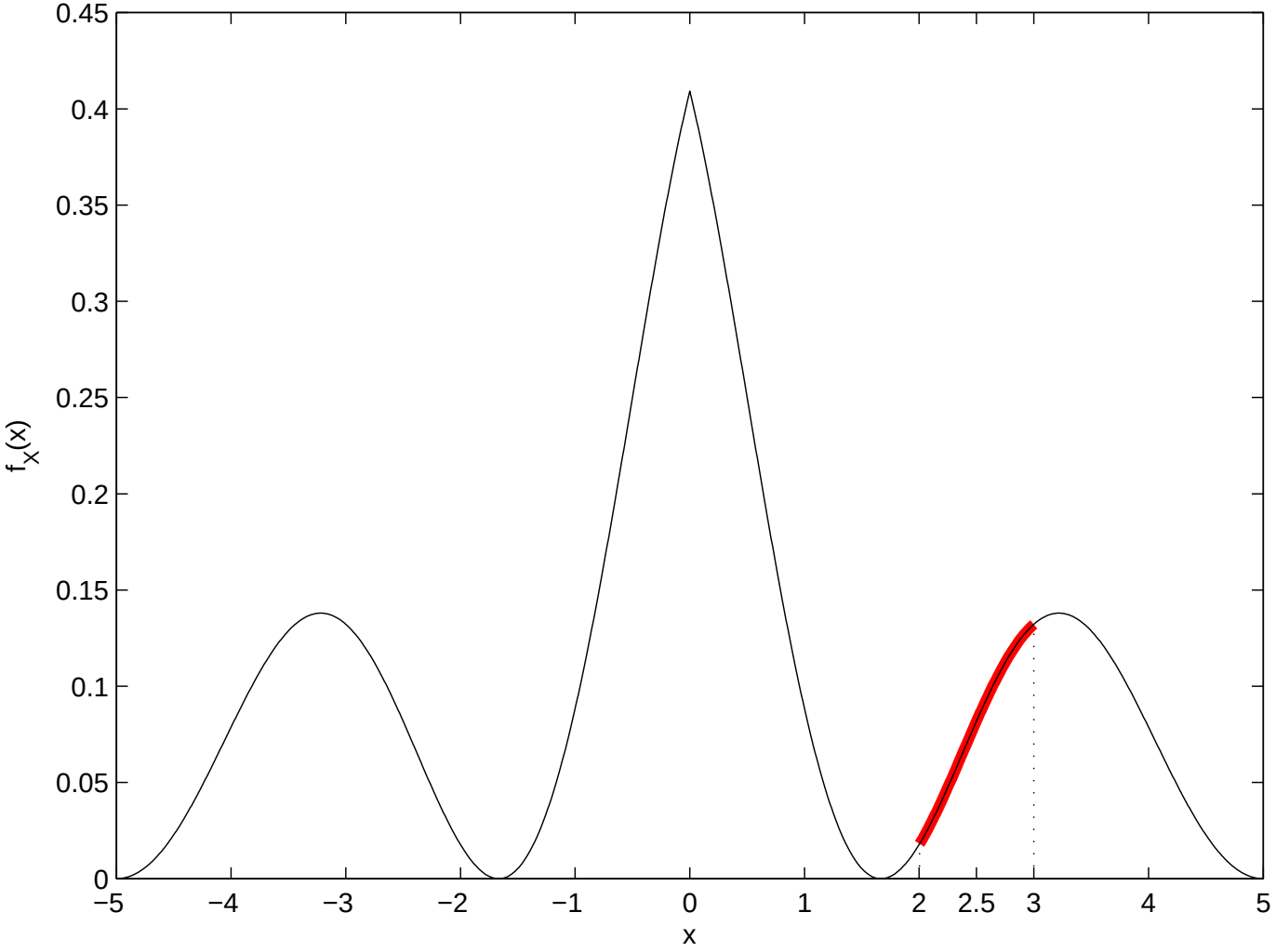
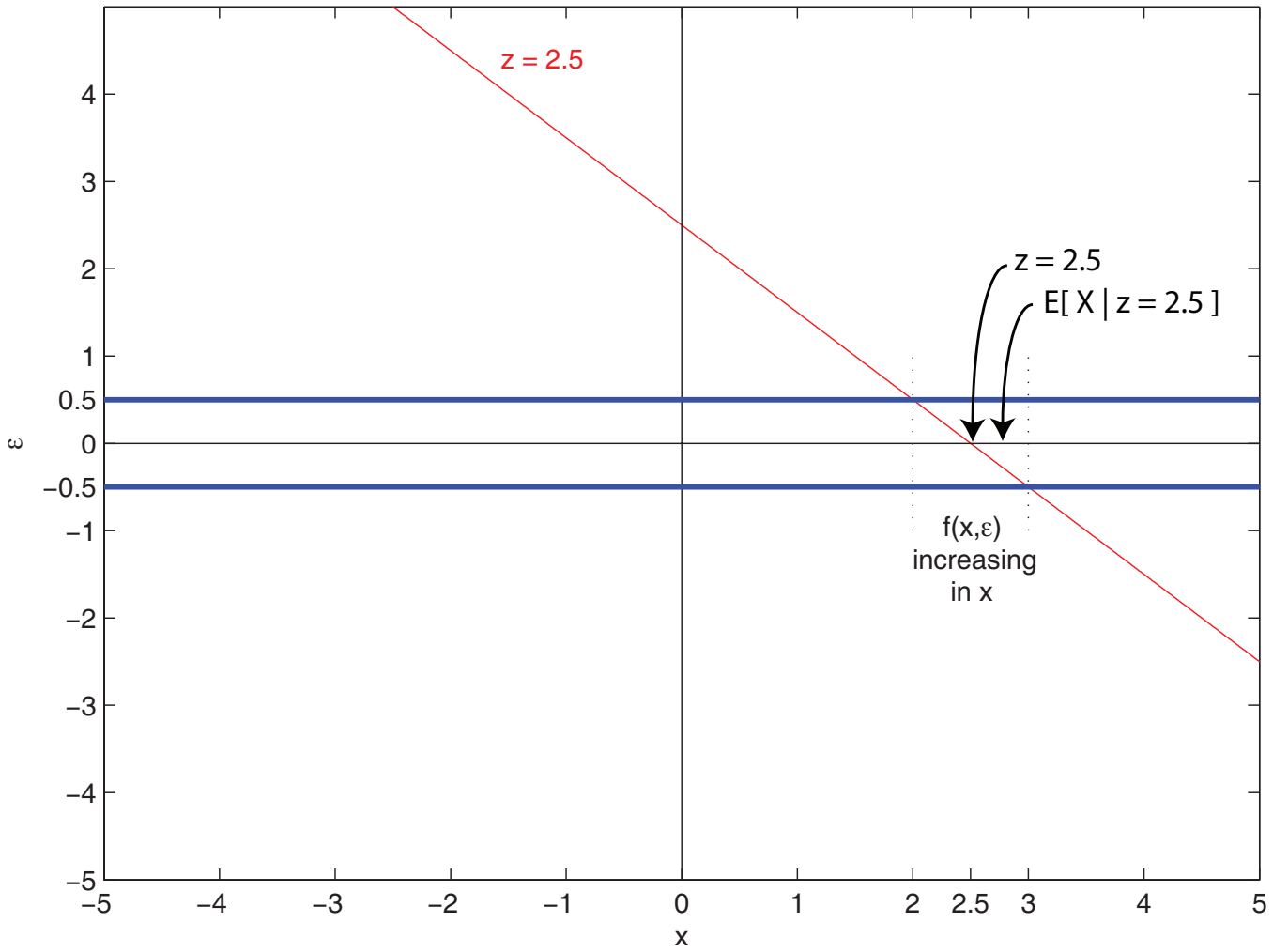


Not quasiconcave: f_X is increasing on $[2, 3]$



If $\tilde{\varepsilon} \sim U[-\frac{1}{2}, \frac{1}{2}]$ then $E[X|z = 2.5] > z = 2.5$. UTS-$\mathcal{E}$ fails.

Summary of Results

Family of Error Terms			Prior			Condition
Sym			Sym		$\Rightarrow$	MR
Sym			Sym		$\not\Rightarrow$	UDS
Sym	QC		Sym		$\Rightarrow$	UDS
Sym	QC		Sym		$\not\Rightarrow$	UTS
Sym	QC	Ind*	Sym	QC	$\Rightarrow$	UTS
	$\mathcal{E}_{2pt} \subseteq \overline{\mathcal{E}}$		Sym		$\Leftarrow$	MR
	$\mathcal{E}_{2pt} \subseteq \overline{\mathcal{E}}$		$\not\exists$		$\Leftarrow$	UDS
	$\mathcal{E}_{U} \subseteq \overline{\mathcal{E}}$		Sym		$\Leftarrow$	MR
	$\mathcal{E}_{U} \subseteq \overline{\mathcal{E}}$		Sym	QC	$\Leftarrow$	UTS

Characterizations

Can form various 'iff' statements:

For $\mathcal{E}_{2pt} \subseteq \overline{\mathcal{E}} \subseteq \mathcal{E}_S$, Sym $X \Leftrightarrow$ MR-$\mathcal{E}$

For $\mathcal{E}_U \subseteq \overline{\mathcal{E}} \subseteq \mathcal{E}_{S,Q}$, Sym $X \Leftrightarrow$ MR-$\mathcal{E} \Leftrightarrow$ UDS-$\mathcal{E}$

For $\mathcal{E}_U \subseteq \overline{\mathcal{E}} \subseteq \mathcal{E}_{S,Q,I}$, Sym & q.-c. $X \Leftrightarrow$ UTS-$\mathcal{E}$

- Bottom Line 1: Strength of updating assumption depends on symmetry and quasiconcavity assumptions on distributions
- Bottom Line 2: Robustness of updating assumption depends on robustness of sym. & q.-c. assumptions on distributions

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