# Elicitability

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# Belief Elicitation via Signals

- **Goal:** elicit subjective $p(E)$ for some event $E \subseteq \Omega$
- **Problem:** states $\omega \in \Omega$ are not observable! Only signals $y \in Y$.

Examples of non-observable states w/ signals:

- Climate change
  - $\omega$: Earth's temp in 100 years. $y$: Earth's temp in 10 years.
- Beliefs in repeated PD w/ private monitoring
  - $\omega$: Actual action ($C$ or $D$). $y$: Observed signal ($c$ or $d$)
- Hot hand belief in basketball
  - $\omega$: correlation parameter. $y$: next shot's outcome
- Linear regression model $y = \beta_0 + \beta_1 x + \varepsilon$
  - $\omega$: parameters $(\beta_0, \beta_1)$. $y$: data $(x_i, y_i)_{i=1}^n$
- Vaccine effectiveness
  - $\omega$: actual effectiveness in $[0, 1]$. $y$: clinical trial in $\{0, 1\}^n$

- **Question:** can we still learn beliefs over $\Omega$ using only $Y$?

# Vaccine Example (of course)

**State:** efficacy. $\theta \in \Theta = \{0, 1/2, 1\}$
**Agent:** medical researcher. Has belief $p \in \Delta(\Theta)$
**Principal:** management. Wants to learn about $p$
**Signal:** outcome of 1 trial. $y \in Y = \{S, H\}$
**Info Structure:** $\Pi(y|\theta)$

| $\Pi$ | | $Y$: Sick (S) | $Y$: Healthy (H) |
|---|---|---|---|
| $\Theta$ | 0 | 1 | 0 |
| | 1/2 | 0.5 | 0.5 |
| | 1 | 0 | 1 |

**Induced Belief on $Y$:** $p_\Pi(S) = \sum_\theta p(\theta)\Pi(S|\theta) = \vec{p} \cdot \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$

**Method:** elicit $p_\Pi$ and try to infer what $p$ must've been

# Vaccine Example: A Tale of Three Agents

| Ann's $p$ | Sick (S) $1/2$ | Healthy (H) $1/2$ |
|---|---|---|
| **0** | 1 | 0 |
| **1** | 0.5 | 0.5 |
| **0** | 0 | 1 |

| Bob's $p$ | $1/2$ | $1/2$ |
|---|---|---|
| **1/2** | 1 | 0 |
| **0** | 0.5 | 0.5 |
| **1/2** | 0 | 1 |

| Charlie's $p$ | $1/2$ | $1/2$ |
|---|---|---|
| **1/3** | 1 | 0 |
| **1/3** | 0.5 | 0.5 |
| **1/3** | 0 | 1 |

# Vaccine Example

| Ann | Bob | $p_\Pi(S)$ | $p_\Pi(H)$ |
|---|---|---|---|
| | | 1 | 0 |
| $\vec{p}$ | $\vec{q}$ | 0.5 | 0.5 |
| | | 0 | 1 |

When are Ann and Bob indistinguishable?

$$p_\Pi(S) = q_\Pi(S)$$

$$\iff$$

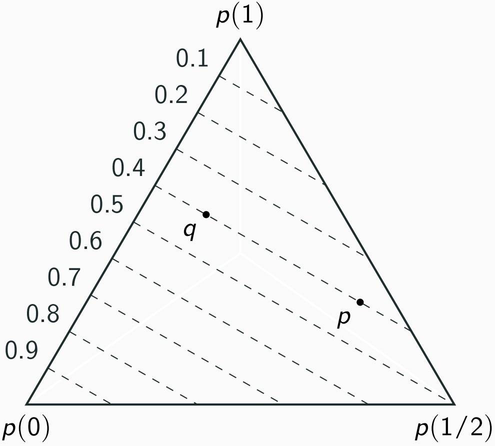
$$\vec{p} \cdot \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \vec{q} \cdot \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

$$\iff$$

$$p(0) + \frac{1}{2}p(1/2) = q(0) + \frac{1}{2}q(1/2)$$

# Vaccine Example

$p_\Pi(S) =$



Given $\Pi$, what can we learn about $p$?

**Main Result:**

$\Pi$ generates a partition of $\Delta(\Theta)$ based on $p_\Pi$.
$p$ and $q$ can be distinguished iff $p_\Pi \neq q_\Pi$

**Two Assumptions:**

1. $p_\Pi$ is derived from $p$ and $\Pi$ via *reduction*
2. $p_\Pi$ can be elicited (BQSR, MPL, ...)

# More General Insight

- Let $Z_y$ be the column associated with each signal $y \in Y$

| | | $Z_S$ | $Z_H$ |
|---|---|---|---|
| | 0 | 1 | 0 |
| $\Theta$ | 1/2 | 0.5 | 0.5 |
| | 1 | 0 | 1 |

- Reduction: $p_{\Pi}(y) = \vec{p} \cdot Z_y$
- So...

$$p_{\Pi}(y) = q_{\Pi}(y) \quad \forall y$$

$$\Longleftrightarrow$$

$$\vec{p} \cdot Z_y = \vec{q} \cdot Z_y \quad \forall y$$

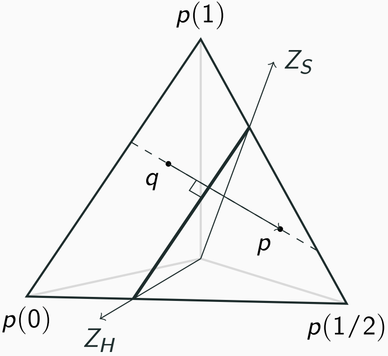
$$\Longleftrightarrow$$

$$(\vec{p} - \vec{q}) \cdot Z_y = 0 \quad \forall y$$

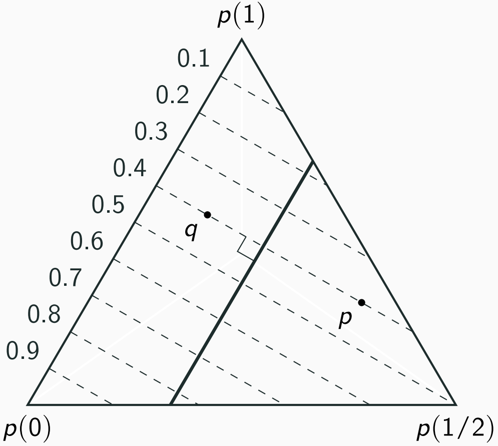
# More General Insight

$$(\vec{p} - \vec{q}) \cdot Z_y = 0 \;\; \forall y$$

"$(\vec{p} - \vec{q})$ is orthogonal to the column span of $\Pi$"



# Vaccine Example



# Vaccine Example: Two Subjects

Now suppose vaccine trial has two patients (iid)
$Y = \{0, 1, 2\}$ gives # of Healthy patients

| | | $Y$: 0 | 1 | 2 |
|---|---|---|---|---|
| | 0 | 1 | 0 | 0 |
| $\Theta$ | 1/2 | 0.25 | 0.50 | 0.25 |
| | 1 | 0 | 0 | 1 |

Three linearly independent columns! $\Pi$ has full rank.
$p_\Pi = \vec{p} \cdot \Pi \Longrightarrow p_\Pi \cdot \Pi^{-1} = \vec{p}$!!

Full rank $\Rightarrow$ We can perfectly back out any belief!

# Random Variables

One observation:

| $\theta$ | $Z_S$ | $Z_H$ |
|---|---|---|
| 0 | 1 | 0 |
| 1/2 | 0.5 | 0.5 |
| 1 | 0 | 1 |

- $Z_H = \theta$
- Elicit $\vec{p} \cdot Z_H = E_p[Z_H]$
- Can learn $E_p[\theta]$

Two observations:

| $\theta$ | $Z_0$ | $Z_1$ | $Z_2$ |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1/2 | 0.25 | 0.50 | 0.25 |
| 1 | 0 | 0 | 1 |

- $Z_2 = \theta^2$ **and** $(Z_2 + \frac{1}{2} Z_1) = \theta$
- Elicit $E_p[Z_2]$ **and** $E_p[Z_1]$
- Can learn $E_p[\theta^2]$ **and** $E_p[\theta]$
  Thus, can learn $Var_p[\theta]$

In general, with $k$ observations, you learn the first $k$ moments of $\theta$

This example: two moments is enough to learn $p$

General: If $|\Theta| = n$ then $n-1$ observations gives you $p$

# The Power of Randomization

"Mixture Experiment"

- 99% chance: use 1 observation. $Y_1 = \{S, H\}$
- 1% chance: use 100 observations. $Y_{100} = \{0, 1, \dots, 100\}$
- Elicit beliefs *before* randomizing

There are now 103 possible signals! $Y = Y_1 \cup Y_{100}$

| | $Y_1$ | | $Y_{100}$ | | | |
|---|---|---|---|---|---|---|
| | $Z_S$ | $Z_H$ | $Z_0$ | $Z_1$ | $\cdots$ | $Z_{100}$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ |
| $\theta$ | $0.99(1-\theta)$ | $0.99\,\theta$ | $0.01(1-\theta)^{100}$ | $1.00\,\theta(1-\theta)^{99}$ | $\cdots$ | $0.01\,\theta^{100}$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ |

Can elicit 100 moments of $\theta$ at a low (expected) cost

# Other Stuff We Know

- Eliciting *median* of $\theta \Leftrightarrow$ you can elicit entire $p$
- Can add covariates
    - $\Pi_{man}$ and $\Pi_{woman}$, $Y = (Y_{man} \times Y_{woman})$
- Infinite states & signals
    - Gaussian linear model: $y = \beta_0 + \beta_1 x + \varepsilon$
        - Full rank! One observation gives entire $p$
    - Non-parametric linear model: $E[y|x] = \beta_0 + \beta_1 x$
        - One obs: $E_p[\beta_0]$, $E_p[\beta_1]$.
        - Two obs: $Var_p[\beta_0]$, $Var_p[\beta_1]$.
        - $\cdots$
    - Probit: $y = \mathbb{1}_{\{\beta_0 + \beta_1 x + \varepsilon > 0\}}$
        - Need *infinite* data to get $E_p[\beta_0]$, $E_p[\beta_1]$!!
- New ordering of Information Structures
    - "$\Pi_2$ elicits more than $\Pi_1$"
    - Blackwell Dominance $\Rightarrow$ Elicits More

# An Experimental Application

Repeated Prisoners' Dilemma with Private Monitoring

- **$j$'s actual action:** $C_j$ or $D_j$
- **$i$'s signal:** $c_j$ or $d_j$
  (actual action will never be shown to $i$)
- **Signal accuracies:** $r > 0.5$ and $s > 0.5$

$\Pi$ | $Y$

| $\Theta$ | | $c_j$ | $d_j$ |
|---|---|---|---|
| | $C_j$ | $r$ | $1 - r$ |
| | $D_j$ | $1 - s$ | $s$ |

- $\Pi$ has full rank since $r \neq 1 - s$
  $\Rightarrow$ can elicit $p_\Pi$ and back out $p$
- Recall: assumes *reduction*

# Summary

- What you can learn about $p$ depends on $\Pi$
    - Full rank $\Rightarrow$ entire $p$ w/ one obs
    - In general, you learn $E_p[Z_y]$ for each column $Z_y$
    - More observations $\Rightarrow$ more $Z_y$ $\Rightarrow$ learn more!
    - Can use mixtures to create more (possible) observations

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