ALTERNATIVE SPECIFICATIONS OF MULTIPLICATIVE RISK PREMIA[†]

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ABSTRACT. In a seminal paper, Pratt [1964] defined a risk premium for multiplicative risks, but did not explore its properties. In the present paper we provide various alternative specifications for multiplicative risk premia. We show how these measures of risk can be used, under varying assumptions, to rank an investor's preferences among multiplicative risks. We find that the multiplicative ask price can be used to rank arbitrary risks for general utility functions. Additionally, we explore an in-depth example highlighting the various results obtained.

Keywords: Risk premium; multiplicative risks.

JEL Classification: D81; G11.

I INTRODUCTION

In a seminal paper, Pratt (1964) defined the risk premium π for additive risks, those of the form $x+Z$. He explored the properties of π and also of the related "bid" and "ask" prices, finding that risks of equal expectation can be ranked according to the values of π. Further, Pratt defined a risk premium for multiplicative risks, but did not explore its properties. In the present paper we give a slightly different definition for a multiplicative risk premium, and also give new definitions of bid and ask prices for multiplicative risks. We show how these measures of risk can be used, under varying assumptions, to rank an investor's preferences among multiplicative risks. Our main result is that the multiplicative ask price can be used to rank arbitrary risks for almost any utility function. We explore an in-depth example highlighting the various results obtained and consider an alternative specification based upon the geometric mean of a risk instead of its expected value.

[†]The authors thank Ivana Komunjer for her helpful comments and conversations.

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II The Model

Definitions

We define the **rate of return** in a given period t to be

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{1}$$

There are several classes of models in the literature which model rates of return in the form[1]

$$R_t = \mu_t(\theta) + \sigma_t(\theta)\varepsilon_t,$$

where $\varepsilon_t \sim iid(0,1)$. Thus shocks affect returns in an additive manner in these models.

We define the **multiplicative risk** associated with the price change in period t to be

$$\begin{aligned} Z_t &= \frac{P_t}{P_{t-1}} \\ &= \exp(R_t) \end{aligned} \tag{2}$$

Since returns are random, we let z represent the realized value of the multiplicative risk Z (omitting time subscripts for notational simplicity). If an investor invests x in some risky asset or portfolio, her final level of wealth given the realized price change will be $x\frac{P_t}{P_{t-1}} = xz$. Thus it would be useful to explore what we can say about an investor's attitudes towards multiplicative risks of the form xZ.

We first define the notion of a multiplicative risk premium. As we show below, the multiplicative risk premium provides a convenient way to characterize an investor's preferences among risks. Specifically, multiplicative risk premia can be used to order an investor's preferences among a certain class of risks. Informally, the multiplicative risk premium of a risk Z is the proportion of $x \cdot E(Z)$ that the investor must give up to make him indifferent between the guaranteed amount the random wealth xZ.

Definition 1. For a given initial wealth x and multiplicative risk Z, the multiplicative risk premium $\pi^m(x,Z)$ is defined by

$$Eu(xZ) = u\left(x\,E(Z)\,\left[1-\pi^m(x,Z)\right]\right) \tag{3}$$

Note that $\pi^m(x,\cdot)$ is well-defined since the argument in the right hand side of 1 ranges over all values of wealth and is monotonic in $\pi^m(\cdot,\cdot)$. We can solve 1 explicitly for $\pi^m(x,Z)$:

$$\pi^m(x,Z) = 1 - \frac{u^{-1}(Eu(xZ))}{xE[Z]}$$

The following definitions conserve language.

Definition 2. A risk Z is called neutral if $E[Z] = 1$.

Definition 3. Two risks Z and Y are called mean-equivalent if $E[Z] = E[Y]$.

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[1] See, for example, Bollserslev (1986, 1987) and McLeaod and Li (1983).

Definition 4. The mapping $(x,Z) \mapsto (xE[Z], Z/E[Z])$ is called the neutralizing transformation.

Note that the neutralizing transformation takes an arbitrary wealth-risk pair and generates a wealth-risk pair with a neutral risk and satisfies $Eu(xZ) = Eu(xE[Z] \cdot Z/E[Z])$.

Results

The following proposition shows that any multiplicative risk Z is equivalent to some neutral risk under the appropriate wealth transformation in the sense that their multiplicative risk premia are equal.

Proposition 1. For any wealth x and risk Z, there exists $\tilde{x}$ and $\tilde{Z}$ such that $E[\tilde{Z}] = 1$ and $\pi^m(x,Z) = \pi^m\left(\tilde{x},\tilde{Z}\right)$.

All proofs are provided in the appendix.

Thus, we can convert generic risks into neutral risks by transforming any risk Z into $Z/E[Z]$ and by multiplying initial wealth x by $E[Z]$.

Pratt **?** defines the proportional risk premium $\pi^*(x,Z)$ by

$$Eu(xZ) = u(x \cdot [E[Z] - \pi^*(x,Z)]). \tag{4}$$

Observe that π^* is not invariant to the neutralizing transformation.

Proposition 2. $\pi^*(x,Z) \neq \pi^*(xE[Z], Z/E[Z])$.

Consequently, we find that $\pi^m(\cdot,\cdot)$ is invariant to the neutralizing transformation while $\pi^*(\cdot,\cdot)$ is not. This invariance is a desirable property of a risk premium as the neutralizing transformation has no affect on the possible outcomes faced by the decision maker.

Note now the relationship between $\pi^*(\cdot,\cdot)$ and $\pi^m(\cdot,\cdot)$.

Proposition 3. For all risks Z, $\pi^*(x,Z) = E[Z]\pi^m(x,Z)$.

This relationship between $\pi^*(\cdot,\cdot)$ and $\pi^m(\cdot,\cdot)$ highlights the intuition behind the fact that $\pi^m(\cdot,\cdot)$ is invariant under the neutralizing transformation while $\pi^*(\cdot,\cdot)$ is not.

Our next result is that mean-equivalent multiplicative risks can be ordered by their multiplicative risk premia $\pi^m(\cdot,\cdot)$.

Theorem 1. For any mean-equivalent risks Z and Y, and at any initial wealth x, $Eu(xZ) > Eu(xY) \Longleftrightarrow \pi^m(x,Z) < \pi^m(x,Y)$.

We next consider the class of utility functions for which multiplicative risk premia are constant in wealth.

Definition 5. $\mathscr{U} = \left\{u : \mathbb{R}^{++} \to \mathbb{R} : forall Z,\ \pi^m(x,Z) = \pi^m(y,Z)\ \forall x,y > 0\right\}$ is the set of utility functions such that multiplicative risk premia π^m are independent of the wealth level.

It turns out that $\mathscr{U}$ coincides with the class of $CRRA$ utility functions which satisfy the condition that $r^*(x) \doteq -\frac{xu''(x)}{u'(x)}$ is constant in x.

Theorem 2. $\mathscr{U} = CRRA = \left\{ u : \mathbb{R}^{++} \to \mathbb{R} : r^*(x) = -\frac{xu''(x)}{u'(x)} is constant in x \right\}$ for relatively small risks.

The above theorem provides a strong result regarding the class of utility functions for which risk premia are constant under wealth, but only applies for sufficiently small risks. By weakening this restriction, we can arrive at a weaker result, though still quite useful.

Theorem 3. $CRRA \subset \mathscr{U}$

It is interesting to note that if $u(x) = \ln(x)$, then $\pi^m(x,Z) = 1 - GM(Z) = 1 - \exp(E[\ln(Z)])$, where $GM(Z)$ is the limit of $\prod_{i=1}^n z_i^{1/n}$ as $n \to \infty$. The implication is that under a certain class of utility functions, the multiplicative risk premium can be calculated simply by considering the geometric mean of a given risk. Furthermore, this implies that within this class of utility functions, mean-equivalent risks can be ranked by their geometric means.

The reader should be careful at this point not to draw erroneous conclusions about the usefulness of $\pi^m(\cdot,\cdot)$ in ranking risks with different means. Take, for example, two arbitrary risks Y and Z with different means and assume $u \in CRRA$. By the neutralizing transformation $\pi^m(x,Y) = \pi^m(xE[Y], Y/E[Y])$. Since $CRRA \subset \mathscr{U}$, then $\pi^m(xE[Y], Y/E[Y]) = \pi^m(x, Y/E[Y])$. By an identical argument, $\pi^m(x,Z) = \pi^m(xE[Z], Z/E[Z]) = \pi^m(x, Z/E[Z])$. Now, since $Y/E[Y]$ and $Z/E[Z]$ have the same mean, we can use $\pi^m(\cdot,\cdot)$ to rank these risks. If, for example, $Eu(xY/E[Y]) > Eu(xZ/E[Z])$, then $\pi^m(x, Y/E[Y]) < \pi^m(x, Z/E[Z])$. Thus, $\pi^m(x,Y) < \pi^m(x,Z)$. While it is therefore true that the risk premium associated with (x,Y) is lower than the risk premium associated with (x,Z), that does not imply $Eu(x,Y) > Eu(x,Z)$. Recall from Theorem 1 that the statement $\pi^m(x,Y) < \pi^m(x,Z) \Rightarrow Eu(x,Y) > Eu(x,Z)$ requires the condition that Y and Z be mean-equivalent. Although the above arguments do guarantee that $\pi^m(x,Y) < \pi^m(x,Z)$, we cannot use this fact to make inferences about the expected utility ranking of the two risks. In fact, counter-examples can be constructed in which $\pi^m(x,Y) < \pi^m(x,Z)$ *and* $Eu(x,Y) < Eu(x,Z)$ when $E[Y] \neq E[Z]$. Such an example is provided in Section III.

After analyzing the properties of $\pi^m(\cdot,\cdot)$, we arrive at the conclusion that its properties are roughly equivalent to those of $\pi^*(\cdot,\cdot)$ with the exception that $\pi^m(\cdot,\cdot)$ is invariant to the neutralizing transformation.

Equivalent & Compensating Risk Premia

We now turn to alternative specifications of risk premia based on the intuition of the "ask" and "bid" premia defined by Pratt with respect to additive risks.

Definition 6. For a given wealth x and risk Z, the equivalent risk premium $\pi_a^m(x,Z)$ is defined by

$$Eu(xZ) = u(x \cdot [1 + \pi_a^m(x,Z)]). \tag{5}$$

Thus the equivalent risk premium is the smallest proportion of wealth the individual would be willing to accept to give up the risk Z.

Definition 7. For a given wealth x and risk Z, the compensating risk premium $\pi_b^m(x,Z)$ is given by

$$u(x) = Eu(x[1 - \pi_b^m(x,Z)] \cdot Z) \tag{6}$$

So the compensating risk premium is the largest proportion of wealth the individual would be willing to pay for the risk Z.

Definition 8. The class of utility functions in which $\pi_a^m(\cdot,Z)$ is constant in wealth is given by $\mathscr{U}_A = \left\{u : \mathbb{R}^{++} \to \mathbb{R} : \pi_a^m(x,Z) = \pi_a^m(y,Z)\ \forall x,y > 0\right\}$.

Proposition 4. If $u \in CRRA$, then $u \in \mathscr{U}_A$.

The next two results state that equivalent and compensating risk premia can be used to order risks in a similar fashion to multiplicative risk premia. Note that the result for the compensating risk premia only applies to mean-equivalent risks and utility functions in the $CRRA$ class, while the equivalent risk premia can order arbitrary risks for any (increasing) utility function.

Theorem 4. For any risks Z and Y, and at any wealth $x > 0$, $Eu(xZ) > Eu(xY) \Longleftrightarrow \pi_a^m(x,Z) > \pi_a^m(x,Y)$.

This theorem applies to all risks, be they mean-equivalent or not. Therefore, $\pi_a^m(\cdot,\cdot)$ is a more general and useful measure for ranking multiplicative risks than is $\pi^m(\cdot,\cdot)$.

Theorem 5. If $u \in \mathscr{U}_A$, then for any risks Z and Y, $Eu(xZ) > Eu(xY) \Longleftrightarrow \pi_b^m(x,Z) > \pi_b^m(x,Y)\ \forall x$.

Remark. Note that the compensating risk premium could be defined by $u(xE[Z]) = Eu\left(xZ\left[1 - \pi_b^m(x,Z)\right]\right)$ in order to gain homogeneity between $\pi_b^m(x,Z)$ and $\pi_a^m(x,Y)$. However, the above theorem that risks can be ranked by $\pi_b^m(x,Z)$ would need the additional assumption of mean-equivalence between the risks being compared for the result to be true.

It is conjectured, but not proven, that $Eu(xZ) > Eu(xY) \Longleftrightarrow \pi_b^m(x,Z) > \pi_b^m(x,Y)\ \forall x$ even if $u \notin \mathscr{U}_A$. The following section provides evidence of this conjecture.

III Example & Discussion

We now provide an in-depth example that highlights the properties of the various risk premia discussed in the previous section. This example also generates a sequence of testable hypotheses that remain unproven.

In this example, consider six different possible risks. These are summarized in the following table.

	L1	L2	N1	N2	H1	H1N
Payoffs	(.2,.7)	(.3,.5,.6)	(.5,1.5)	(.75,1.1)	(2,12)	(.2,1.2)
Probabilities	(.6,.4)	(.6,.2,.2)	(.5,.5)	$\left(\frac{10}{35},\frac{25}{35}\right)$	(.2,.8)	(.2,.8)
$E[Payoff]$	0.4	0.4	1	1	10	1

$L1$ and $L2$ are mean-equivalent with expectations of 0.4. $N1$ and $N2$ are neutral risks with different variances. $H1$ is a very favorable risk and $H1N$ is the neutralized version of $H1$. We now consider the values of $Eu(xZ)$, $\pi^m(x,Z)$, $\pi^*(x,Z)$, $\pi_a^m(x,Z)$, and $\pi_b^m(x,Z)$ for each of the above risks when (1) $x=1$ and $u \in CRRA$, (2) $x=1$ and $u \notin CRRA$, (3) $x=10$ and $u \in CRRA$, and (4) $x=10$ and $u \notin CRRA$. The results are as follows.

$x=1 \quad u(x)=\sqrt{x}$						$(x=10)$
	L1	L2	N1	N2	H1	H1N
$Eu(xZ)$	0.603	0.625	0.966	0.997	3.054	3.054
Ranking	6	5	4	3	1	1
$\pi^m(x,Z)$	0.091	0.024	0.067	0.007	0.0672	0.0672
Ranking	6	2	3	1	4	4
$\pi^*(x,Z)$	0.0364	0.0096	0.067	0.007	0.672	0.0672
Ranking	3	2	4	1	6	5
$\pi_a^m(x,Z)$	-0.636	-0.609	-0.067	-0.007	8.328	-0672
Ranking	5	4	3	2	1	6
$\pi_b^m(x,Z)$	-1.75	-1.56	-0.072	-0.00687	0.893	-0.0721
Ranking	6	5	3	2	1	4

$x=1 \quad u(x)=1-e^{-x}$						$(x=10)$
	L1	L2	N1	N2	H1	H1N
$Eu(xZ)$	0.31	0.324	0.585	0.627	0.973	0.973
Ranking	6	5	4	3	1	1
$\pi^m(x,Z)$	0.0719	0.0195	0.1201	0.0131	0.6391	0.6391
Ranking	3	2	4	1	5	5
$\pi^*(x,Z)$	0.02876	0.0078	0.1201	0.0131	6.391	0.6391
Ranking	3	1	4	2	5	5
$\pi_a^m(x,Z)$	-0.629	-0.608	-0.12	-0.013	0.261	-0.6391
Ranking	5	4	3	2	1	6
$\pi_b^m(x,Z)$	-2.106	-1.628	-0.159	-0.0135	0.888	-3.195
Ranking	5	4	3	2	1	6

$x = 10 \quad u(x) = \sqrt{x}$						$(x = 100)$
	L1	L2	N1	N2	H1	H1N
$Eu(xZ)$	1.907	1.976	3.055	3.151	9.658	9.658
Ranking	6	5	4	3	1	1
$\pi^m(x,Z)$	0.091	0.024	0.067	0.007	0.0672	0.0672
Ranking	6	2	3	1	4	4
$\pi^*(x,Z)$	0.0364	0.0096	0.067	0.007	0.672	0.0672
Ranking	3	2	4	1	6	5
$\pi_a^m(x,Z)$	-0.636	-0.609	-0.67	-0.007	8.328	-0.0672
Ranking	5	4	3	2	1	6
$\pi_b^m(x,Z)$	-1.75	-1.56	-0.072	-0.00687	0.893	-0.0721
Ranking	6	5	3	2	1	4

$x = 10 \quad u(x) = 1 - e^{-x}$						$(x = 100)$
	L1	L2	N1	N2	H1	H1N
$Eu(xZ)$	0.918	0.968	0.997	0.999	1	1
Ranking	6	5	4	3	1	1
$\pi^m(x,Z)$	0.373	0.137	0.431	0.132	0.784	0.784
Ranking	3	2	4	1	5	5
$\pi^*(x,Z)$	0.1492	0.0548	0.431	0.132	7.84	0.784
Ranking	3	1	4	2	5	5
$\pi_a^m(x,Z)$	-0.749	-0.655	-0.431	-0.132	1.161	0
Ranking	6	5	4	3	1	2
$\pi_b^m(x,Z)$	-3.744	-2.163	-0.858	-0.17	0.579	0
Ranking	6	5	4	3	1	2

This example verifies several properties derived in the previous section of this paper. For example, $\pi^m(x,Z)$ appropriately ranks all mean-equivalent risks. This can be seen by comparing the ranking of any set of mean-equivalent risks at the same wealth level. Similarly, we see that $\pi^*(x,Z)$ ranks all mean-equivalent risks, but does not agree with $\pi^m(x,Z)$ in ranking across wealth levels or across risks of different expectation. This gives evidence of the inability of either measure to rank risks across different wealth levels or provide a general ranking of risks regardless of their expectation. Note also that $\pi^*(1,H1) \neq \pi^*(10,H1N)$, which verifies the conclusion that $\pi^*(\cdot,\cdot)$ is not invariant under the neutralizing transformation.

We do observe that $\pi_a^m(x,Z)$ ranks all risks of the same wealth levels identically to the expected utility rankings in all scenarios, as does $\pi_b^m(x,Z)$. Recall our conjecture that the assumption $u \in CRRA$ is not needed for $\pi_b^m(x,Z)$ to rank arbitrary risks. Note that the $H1N$ column in each table is taken at a different wealth level than the rest of the table, so its ranking should be discarded in this comparison. However, $H1N$ for $x = 1$ can be compared to the $x = 10$ table. This is particularly useful in highlighting the inability of $\pi^m(x,Z)$ in ranking risks at different wealth levels even though $u \in \mathscr{U}$. For example, we have

$$0.0672 = \pi^m(1,H1) = \pi^m(10,H1N) > \pi^m(10,N1) = \pi^m(1,N1) = 0.067$$

while $3.054 = Eu(1,H1) > Eu(1,N1) = 0.966$. So, even though we can guarantee that $\pi^m(1,H1) > \pi^m(1,N1)$ by the properties of $\pi^m(\cdot,\cdot)$ when $u \in \mathscr{U}$, we cannot make inferences about expected utility based on these rankings. It is instead useful to consider the ranking of $\pi^m_a(\cdot,\cdot)$ as it corresponds exactly with $Eu(\cdot)$ when $u \in \mathscr{U}$.

We find that for $u \in \mathscr{U}$, $\pi^m(x,Z)$ is invariant to the change in wealth and that this is not necessarily true outside of $\mathscr{U}$. Note that $\pi^*(x,Z)$ is not invariant to wealth within $\mathscr{U}$. Although this was not directly shown, its proof is similar to the proof used in the $\pi^m(x,Z)$ case.

From this example, we note that the absolute rankings based on $\pi^m(\cdot,\cdot)$ and $\pi^*(\cdot,\cdot)$ appear to be invariant to wealth regardless of whether or not $u \in \mathscr{U}$. This conjecture is not proven in our paper. Furthermore, $\pi^m_b(\cdot,\cdot)$ appropriately ranked risks in this example for $u \notin \mathscr{U}$ even though this remains to be proven or disproved. Finally, the example implies that $\pi^m_b(\cdot,\cdot)$ may be invariant to wealth changes when $u \in \mathscr{U}$ although this is yet to be shown.

IV Geometric Means-Based Multiplicative Risk Premia

The development above defines risk premia based on the proportion of wealth that an individual would be willing to sacrifice to be indifferent between a risk and its expected value. An alternative method can be used to generate risk premia based on the proportion of wealth an individual would sacrifice to be indifferent between a risk and its geometric mean. The reader can verify that the properties of this alternative specification for risk premia are the same as those given above for risk premia in terms of expectations.

We first give two definitions.

Definition 9. The sample geometric mean of a sample $z = z_1, z_2, ..., z_n$ drawn $i.i.d.$ from a random variable Z is defined as

$$SGM_n(z) = \prod_{i=1}^{n} z_i^{1/n} \tag{7}$$

Definition 10. The geometric mean of a random variable Z with $c.d.f.$ F_Z is defined as

$$\begin{aligned} GM(Z) &= \exp\left(\int \ln(z) dF_Z(z)\right) \\ &= \exp(E[\ln(Z)]) \end{aligned} \tag{8}$$

We then have the result that $SGM_n(z)$ converges to $GM(Z)$.

Proposition 5. Given $i.i.d.$ samples z from a random variable Z, as $n \to \infty$, $SGM_n(z) \to GM(Z)$ almost surely.

We next define the geometric risk premium for the geometric mean.

Definition 11. For a given initial wealth x and multiplicative risk Z, the geometric risk premium $\gamma^m(x,Z)$ is defined by

$$Eu(xZ) = u\left(x\,GM(Z)\,\left[1-\gamma^m(x,Z)\right]\right) \tag{9}$$

We show only the analogous result to theorem (1), claiming that the other results follow in a similar fashion. We call a multiplicative risk Z GM-neutral if $GM(Z) = 1$.

The following proposition shows that any multiplicative risk Z is equivalent to a GM-neutral risk in the sense that their geometric risk premia are equal.

Proposition 6. For any multiplicative risk Z, $\gamma^m(x,Z) = \gamma^m(x \cdot GM(Z), Z/GM(Z))$.

The proof of this proposition is similar to that of the corresponding proposition in the Section , and is therefore excluded.

Theorem 6. For any GM-equivalent risks Z and Y, and at any wealth $x > 0$, $Eu(xZ) > Eu(xY) \Longleftrightarrow \gamma^m(x,Z) < \gamma^m(x,Y)\ \forall x$.

The geometric means approach provides an interesting alternative to the usual setup, but gains us nothing in the way of desirable properties of risk premia or model tractability. Furthermore, it can be shown that $\gamma^m(\cdot,Z) \neq 0$ for risk neutral decision makers when $GM(Z) = 1$, whereas $\pi^m(\cdot,Z) = 0$ for risk-neutral decision makers when $E[Z] = 1$. Thus, we conclude that $\pi^m(\cdot,\cdot)$ and its related measures are preferable to their geometric mean counterparts.

V Concluding Remarks

We have defined various alternative specifications for multiplicative risk premia. Each specification is defined using a different intuition. For example, $\pi_a^m(\cdot,\cdot)$ is specified to quantify the proportion of wealth a decision maker would accept to sell a given risk. Each has its own properties with respect to ranking risks. We find that the general risk premia $\pi^m(\cdot,\cdot)$ and $\pi^*(\cdot,\cdot)$ which are most commonly used lack the more general properties of the compensating premium $\pi_a^m(\cdot,\cdot)$. The compensating premium is useful in ranking all risks at a given initial wealth level. Although not proven, it is conjectured that $\pi_b^m(\cdot,\cdot)$ has similar usefulness. Given the intuition behind the construction of these risk premia, it is perhaps not surprising that they in fact rank risks so well. $\pi_a^m(\cdot,\cdot)$ and $\pi_b^m(\cdot,\cdot)$ are essentially the supply and demand of various risks, which indicate the amount a decision maker values a given risk. Consequently, we find these measures to be more desirable gauges of a decision maker's preferences.

We briefly explore the merits of risk premia based on the geometric mean of a risk rather than its expectation. The reasoning behind this exploration is that perhaps multiplicative risks are better compared using a multiplicative measure of center rather than an additive measure such as the expectation. However, we find no benefit to such risk premia and in fact find undesirable properties of the premium with respect to risk neutral decision makers. Therefore, we conclude that $\pi_a^m(\cdot,\cdot)$ and perhaps $\pi_b^m(\cdot,\cdot)$ are the appropriate risk premia to be used when comparing various risks.

APPENDIX

Proof of Proposition 1

We first note that

$$Eu(xZ) = Eu\left(x\mu\left[Z/\mu\right]\right) \tag{10}$$

for any constant $\mu > 0$. In particular, consider $\mu = E[Z]$, which gives the neutralizing transformation.

Using Definition 1, we have that

$$u\left(x\,E[Z]\,\left[1-\pi^m(x,Z)\right]\right) = u\left([x\,E[Z]]\,[E[Z/E[Z]]\,]\,\left[1-\,\pi^m(xE[Z],Z/E[Z])\right]\right) \tag{11}$$

Note that for any random variable X, $E[X/E[X]] = 1$. Therefore

$$u\left(x\,E[Z]\,\left[1-\pi^m(x,Z)\right]\right) = u\left([x\,E[Z]]\left[1-\,\pi^m(xE[Z],Z/E[Z])\right]\right) \tag{12}$$

Under the assumption of $u'(x) > 0$ for all $x > 0$, we have that

$$\begin{aligned} [x\,E[Z]]\,\left[1-\pi^m(x,Z)\right] &= [x\,E[Z]]\,\left[1-\pi^m(xE[Z],Z/E[Z])\right] \\ \pi^m(x,Z) &= \pi^m(xE[Z],Z/E[Z]). \end{aligned} \tag{13}$$

which proves the assertion.

Q.E.D.

Proof of Proposition 2

The following are equivalent.

$$\begin{aligned} Eu(xZ) &= Eu((xE[Z]\cdot Z/E[Z]) \\ u(x\left[E[Z]-\pi^*(x,Z)\right] &= u(xE[Z]\left[1-\pi^*(xE[Z],Z/E[Z])\right]) \\ E[Z]-\pi^*(x,Z) &= E[Z]\left[1-\pi^*(xE[Z],Z/E[Z])\right] \\ E[Z]\pi^*(xE[Z],Z/E[Z]) &= \pi^*(x,Z) \end{aligned} \tag{14}$$

But then $\pi^*(xE[Z],Z/E[Z]) = \pi^*(x,Z) \Leftrightarrow E[Z] = 1$, which is not true in general.

Q.E.D.

Proof of Proposition 3

The following are equivalent.

$$\begin{aligned} Eu(xZ) &= Eu(xZ) \\ u\left(xE[Z]\left(1-\pi^m(x,Z)\right)\right) &= u\left(x\cdot\left[E[Z]-\pi^*(x,Z)\right]\right) \\ E[Z]-E[Z]\pi^m(x,Z) &= E[Z]-\pi^*(x,Z) \\ E[Z]\pi^m(x,Z) &= \pi^*(x,Z) \end{aligned} \tag{15}$$

Q.E.D.

Proof of Theorem 1

The following statements are all equivalent assuming $u'(x) > 0$ for all $x > 0$:

$$\begin{aligned} Eu(xZ) &> Eu(xY) \\ u\left(x\,E[Z]\,\left[1-\pi^m(x,Z)\right]\right) &> u\left(x\,E[Y]\,\left[1-\pi^m(x,Y)\right]\right) \\ x\,E[Z]\,\left[1-\pi^m(x,Z)\right] &> x\,E[Y]\,\left[1-\pi^m(x,Y)\right] \\ \pi^m(x,Z) &< \pi^m(x,Y) \end{aligned} \tag{16}$$

Q.E.D.

Proof of Theorem 2

As in Pratt [64], we have that $\pi^*(x,Z) = \frac{1}{2}\sigma_Z^2 r^*(x) + o\left(\sigma_Z^2\right)$ for small risks Z. Thus, if $r^*(x)$ is constant in x, so too is $\pi^*(x,Z)$. By Proposition 3, this implies that $\pi^m(x,Z)$ is also constant in x. By the same reasoning, if $r^*(x)$ is not constant in x, then neither is $\pi^m(x,Z)$.

Q.E.D.

Proof of Theorem 3

We know from Pratt **?** that

$$CRRA = \begin{cases} a\ln x + b,\ a > 0, b \in \mathbb{R},\ \rho = 1 \\ ax^{\rho} + b,\ a > 0, b \in \mathbb{R},\ \rho < 0 \\ -ax^{-\rho} + b,\ a > 0, b \in \mathbb{R},\ \rho > 0,\ \rho \neq 1 \end{cases} \tag{17}$$

(i) Take $u(x) = a\ln x + b$. $\pi^m(x,Z) = 1 - \frac{u^{-1}(Eu(xZ))}{xE[Z]}$, so it suffices to show that $\frac{u^{-1}(Eu(xZ))}{xE[Z]} = k$, for some variable k that is independent of x, where $u^{-1}(y) = \exp(\frac{y-b}{a})$. We have

$$\begin{aligned} \frac{\exp\left(\frac{\int a\ln(xz) + b dF_Z(z) - b}{a}\right)}{x\int z dF_Z(z)} &= k \\ \frac{\exp\left[\int \ln(xZ)\, dF_Z(z)\right]}{x\int z dF_Z(z)} &= k \\ \int \ln(xZ)\, dF_Z(z) &= \ln k + \ln \int xZ dF_Z(z) \\ \exp(E[\ln(Z)]) &= k, \end{aligned} \tag{18}$$

which implies that k is indeed independent of x.

(ii) Take $u(x) = ax^\rho + b$, so that $u^{-1}(y) = \left(\frac{y-b}{a}\right)^{1/\rho}$. Solving for $\pi^m(\cdot,\cdot)$ as in the previous step, we have

$$\frac{\left(\frac{\int [ax^\rho z^\rho + b] dF_Z(z) - b}{a}\right)^{1/\rho}}{xE[Z]} = k$$
$$\left(\int x^\rho z^\rho dF_Z(z)\right)^{1/\rho} = kxE[Z]$$
$$\left(\int z^\rho dF_Z(z)\right)^{1/\rho} = kE[Z]$$
$$\left(\frac{E[Z^\rho]}{E[Z]^\rho}\right)^{1/\rho} = k$$

which implies that k is independent of x. Thus, $\pi^m(\cdot,\cdot)$ is independent of x for $u(x) = ax^\rho + b$.

(iii) The proof is similar to part (ii).

Since $\pi^m(\cdot, Z)$ is constant in x for all $u \in CRRA$, we have that $CRRA \subset \mathscr{U}$.

Q.E.D.

Proof of Theorem 4

Observe that $\pi_a^m(x,Z) = \frac{u^{-1}(Eu(xZ))}{x} - 1$. Using the same logic that was used to show $CRRA \subset \mathscr{U}$, we conclude that $CRRA \subset \mathscr{U}_A$.

Q.E.D.

Proof of Theorem 4

Assuming $u'(x) > 0$ for all $x > 0$, the following statements are all equivalent:

$$Eu(xZ) > Eu(xY)$$
$$u(x \cdot [1 + \pi_a^m(x,Z)]) > u(x \cdot [1 + \pi_a^m(x,Y)]) \tag{19}$$
$$x \cdot [1 + \pi_a^m(x,Z)] > x \cdot [1 + \pi_a^m(x,Y)]$$
$$\pi_a^m(x,Z) > \pi_a^m(x,Y).$$

Q.E.D.

Proof of Theorem 5

We have from the definition of $\pi_b^m(\cdot,\cdot)$ that

$$Eu(x[1 - \pi_b^m(x,Z)]Z) = u(x) = Eu(x[1 - \pi_b^m(x,Y)]Y) \tag{20}$$

or

$$Eu(w_Z Z) = Eu(w_Y Y) \tag{21}$$

with $w_Z = x[1-\pi_b^m(x,Z)]$, $w_Y = x[1-\pi_b^m(x,Y)]$. Then by the definition of $\pi^m(\cdot,\cdot)$, we have

$$\begin{aligned} & u(w_Z(1+\pi_a^m(w_Z,Z))) = u(w_Y(1+\pi_a^m(w_Y,Y))) \\ & \Leftrightarrow \\ & w_Z(1+\pi_a^m(w_Z,Z)) = w_Y(1+\pi_a^m(w_Y,Y)). \end{aligned} \tag{22}$$

This last line implies that $w_Z < w_Y \Leftrightarrow \pi_a^m(w_Z,Z) > \pi_a^m(w_Y,Y)$.

Assuming that $u \in \mathscr{U}_A$, we have the following equivalent statements.

$$\begin{aligned} & \pi_b^m(x,Z) > \pi_b^m(x,Y) \\ & w_Z < w_Y \\ & \pi_a^m(w_Z,Z) > \pi_a^m(w_Y,Y) \\ & \pi_a^m(x,Z) > \pi_a^m(x,Y) \\ & Eu(xZ) > Eu(xY) \end{aligned} \tag{23}$$

Q.E.D.

Proof of Theorem 5

We have

$$\begin{aligned} SGM_n(z) &= \prod_{i=1}^{n} z_i^{1/n} \\ &= \exp(\frac{1}{n}\sum_{i=1}^{n} \ln(z_i)). \end{aligned} \tag{24}$$

By the strong law of large numbers this last quantity converges almost surely to $\exp(E[\ln Z])$, and we are done.

Q.E.D.

Proof of Theorem 6

The following statements are all equivalent assuming $u'(x) > 0$ for all $x > 0$:

$$\begin{aligned} & Eu(xZ) > E(xY) \\ & u\left(x\, GM(Z)\ \left[1-\gamma^m(x,Z)\right]\right) > u\left(x\, GM(Y)\ \left[1-\gamma^m(x,Y)\right]\right) \\ & x\, GM(Z)\ \left[1-\gamma^m(x,Z)\right] > x\, GM(Y)\ \left[1-\gamma^m(x,Y)\right] \\ & \gamma^m(x,Z) < \gamma^m(x,Y) \end{aligned} \tag{25}$$

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