Designing Stable Mechanisms for Economic Environments

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Abstract

Past experimental results suggest that dynamic stability is an important property of mechanisms in public goods settings. We show how to design mechanisms that not only implement the Walrasian or Lindahl equilibrium allocations, but also induce contraction mappings for a wide range of quasi-linear economies. We do this in three steps: First, we identify strong necessary conditions on the functional form of any mechanism that implements Walrasian or Lindahl equilibria. Second, we use these necessary conditions to identify impossibility results for mechanisms with small strategy spaces. Finally, we show how to use additional dimensions to construct contractive mechanisms.

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1 Introduction

To be written.

2 The Model

2.1 Economic Environments

Consider an economy that consists of n agents each endowed with a preference relation defined on a two-dimensional commodity space. For each trader i denote the individually feasible consumption set by $\mathcal{C}_i$ and the endowment vector by $\omega_i = (\omega_x^i, \omega_y^i)$. We require that $\omega_i \in \mathcal{C}_i$ for each i. The two-dimensional net trade vector of agent i is given by $z_i = (x_i, y_i)$ and the set of i's individually feasible net trades is $\mathcal{Z}_i := \mathcal{C}_i - \{\omega_i\}$. Thus, we can describe i's preferences over net trades by a preference relation defined on $\mathcal{Z}_i$. We assume that, for all $i \in \mathcal{I}$, if $z \in \mathcal{Z}_i$ and $z' \geq z$ then $z' \in \mathcal{Z}_i$. A net trade vector $z = (z_1, \ldots, z_n)$ is said to be *feasible* if $z_i \in \mathcal{Z}_i$ for each i and *balanced* if $\sum_i z_i = 0$.

For simplicity we assume that each agent's preferences over net trades are representable by a utility function of the form $u_i(x_i, y_i|\theta_i)$, where θ_i identifies i's *type* and is drawn from some set of admissible types Θ_i. Define $\Theta = \times_i \Theta_i$ to be the set of all admissible type profiles. We assume throughout that u_i is strictly increasing in x_i (the numéraire good) for all y_i and θ_i; when we describe results on stability we further assume that u_i is quasilinear in x_i. We let p represent the price of the second good, normalizing the numéraire price to one.

As specified, the model describes an exchange economy with purely private goods. But we can easily reinterpret the model to allow the second good to be a purely public good by making three changes: (1) every feasible net trade must be such that $y_i = y_j$ for all agents i and j, (2) $\omega_y^i = \omega_y^j$ for all i and j, (3) there is a firm which produces y units of the public good from $c(y)$ units of the numéraire and aims to maximize profit $(py - c(y))$; for simplicity, we assume a constant marginal cost of production: $c(y) = \kappa y$ for all y, with $\kappa > 0$, and (4) the balance condition becomes $\kappa y + \sum_i x_i = 0$.

A Walrasian equilibrium of a private goods economy is a net trade vector z^* and a price p^* such that z^* is balanced, feasible, and maximizes each agent i's utility among all feasible net trades satisfying i's budget constraint that $x_i + p^* y_i \leq 0$.

A Lindahl equilibrium of a public goods economy is a net trade vector z^* and a vector of individual prices $p^* = (p_1^*, \ldots, p_n^*)$ such that z^* is balanced, feasible (which guarantees that $y_i^* = y_j^* = y^*$ for all i and j), maximizes each agent i's utility among all feasible net trades satisfying i's budget constraint that $x_i + p_i^* y_i \leq 0$, and, among all feasible net trades, maximizes the firm's profit of $(\sum_i p_i^*)y - c(y)$.

Note that Lindahl equilibria are of the same dimensionality as Walrasian equilibria; the latter requires $2n$ quantities but only one price while the former requires only $n+1$ quantities but needs n prices.

To avoid issues with boundary equilibria, we assume that $\mathcal{C}_i = \mathbb{R}^2$, so that all net trades are individually feasible, and that u_i is strictly concave in y_i for all i at all $\theta \in \Theta$. These assumptions force every Walrasian or Lindahl equilibrium to be an interior equilibrium.

2.2 Mechanisms & Implementation

A *social choice correspondence* $f : \Theta \twoheadrightarrow \mathcal{Z}$ maps type profiles into sets of allocations. For example, f might identify all Pareto optimal net trades for each θ (the *Pareto correspondence*), all net trades z for which there is some price p such that (z, p) constitutes a Walrasian equilibrium at θ (the *Walrasian correspondence*), or, in a public goods setting, all net trades z for which there is some price vector p such that (z, p) constitutes a Lindahl equilibrium at θ (the *Lindahl correspondence*).

A *mechanism* $\Gamma = (\mathcal{M}, h)$ is a *message space* $\mathcal{M} = \times_i \mathcal{M}_i$ and an *outcome function* $h : \mathcal{M} \twoheadrightarrow \mathcal{Z}$ mapping each message profile $m = (m_1, \ldots, m_n)$ into a net trade vector z. A Nash equilibrium message profile of the mechanism Γ at the type profile θ is an $m^* \in \mathcal{M}$ such that, for each i and $m_i' \in \mathcal{M}_i$,

$$u_i(h(m^*)|\theta_i) \geq u_i(h(m_i', m_{-i}^*)|\theta_i),$$

where (m_i', m_{-i}^*) represents the message vector where i chooses m_i' and each $j \neq i$ chooses m_j^*. We write

$$U_i(m|\theta_i) := u_i(h(m)|\theta_i)$$

to describe i's preferences over strategies in the mechanism Γ given environment θ. Thus, at each θ, Γ induces a normal form game given by $(\mathcal{I}, \mathcal{M}, (U_i(\cdot|\theta_i))_i)$. The *Nash correspondence* $\nu : \Theta \twoheadrightarrow \mathcal{M}$ identifies the set of pure-strategy Nash equilibrium message profiles for each induced game $(\mathcal{I}, \mathcal{M}, (U_i(\cdot|\theta_i))_i)$. A mechanism $(\mathcal{M}, h)$ is said to *implement* a social choice correspondence f if, for all $\theta \in \Theta$,

$$h(\nu(\theta)) = f(\theta).$$

In the case of economic environments with two goods, the outcome function can equivalently be written as a vector of $2n$ functions of the form $x_i(m)$ and $y_i(m)$ for each $i \in \mathcal{I}$.

In the most general form, a mechanism for implementing Walrasian or Lindahl equilibria can be specified by a mapping

$$\mathcal{M} \ni m \mapsto (x_i(m), y_i(m))_{i=1}^n \in \mathcal{Z}. \tag{1}$$

When $\mathcal{M}_i$ has dimensions that do not enter into y_i we may, for notation's sake, partition agent i's strategy space into $\mathcal{M}_i = \mathcal{R}_i \times \mathcal{S}_i$ with $\mathcal{R}_i \subseteq \mathbb{R}^{J_i}$ and $\mathcal{S}_i \subseteq \mathbb{R}^{K_i - J_i}$. Letting $\mathcal{R} = \times_i \mathcal{R}_i$ and $\mathcal{S} = \times_i \mathcal{S}_i$, we have that $y_i : \mathcal{R} \to \mathbb{R}$ and $x_i : \mathcal{R} \times \mathcal{S} \to \mathbb{R}$.

Given any mechanism with functions $y_i(m)$, it is without loss of generality that we can express

i's net trade of the numéraire as

$$x_i(m) = -q_i(m_{-i})y_i(m) - g_i(m) \tag{2}$$

so that the per-unit 'price' term q_i does not depend on m_i (and could be identically equal to zero) and the 'penalty' term $g_i(m)$ is arbitrary. Thus, any mechanism can be equivalently described by the mapping $m \mapsto (y_i(m), q_i(m_{-i}), g_i(m))_{i=1}^n$. It is with this formulation with which we will proceed.

3 Stability, Supermodularity, and Contractive Mechanisms

3.1 Instability in Games with Monotone Best-Responses

Consider the game $(\mathcal{I}, \mathcal{M}, (U_i)_i)$ induced by some mechanism. If each $\mathcal{M}_i$ is a subset of $\mathcal{R}^{K_i}$ and if each U_i is twice differentiable everywhere then, following Milgrom and Roberts (1990), this game is said to be *supermodular* if

1. $\partial^2 U_i / \partial m_{ik} \partial m_{il} \geq 0$ for all $i \in \mathcal{I}$ and $k \neq l \in \{1, \ldots, K_i\}$,
2. $\partial^2 U_i / \partial m_{ik} \partial m_{jl} \geq 0$ for all $i \neq j \in \mathcal{I}$, $k \in \{1, \ldots, K_i\}$, and $l \in \{1, \ldots, K_j\}$, and
3. $\mathcal{M}_i$ is a compact set in $\mathbb{R}^{K_i}$ for all i.

Milgrom and Roberts (1990) then prove that for every supermodular game there is a smallest and largest Nash equilibrium, denoted here by $\underline{m}_i^*$ and $\overline{m}_i^*$. Furthermore, if a learning dynamic is 'adaptive'—roughly, if it selects undominated strategies against a not-too distant history of past play—then that dynamic will converge to the interval $[\underline{m}_i^*, \overline{m}_i^*]$; if, in fact, the game has a unique equilibrium, then this interval is degenerate and the unique equilibrium is globally stable under all adaptive dynamics.

Almost all of the existing literature on stability in mechanism design for economic environments has focused on supermodularity as a sufficient condition for global stability of a mechanism Chen (2002); Mathevet (2007); Chen (2008). This is because its requirements are fairly easy to verify and—more importantly—because existing experimental research seems to corroborate the strong theoretical properties: If a mechanism induces a supermodular game, then subjects' behavior in fact converges to the equilibrium points, and in many mechanisms that are not supermodular subjects' behavior does not converge (Chen and Plott, 1996; Chen and Tang, 1998; Healy, 2006, e.g.).

Unfortunately, there is a disconnect between the theoretical results on the dynamic stability of supermodular games and the ways in which that theory has often been applied to mechanism design. Specifically, the first two requirements of supermodularity listed above are always applied, but the requirement that the strategy space be a closed interval has been regularly ignored.[1] Applying Milgrom & Roberts's results to such a game can lead to incorrect conclusions about a game's stability properties.

To illustrate, consider a simple two-player game with $\mathcal{M}_i = [-100, 100]$ for $i \in \{1, 2\}$ and

$$U_i(m_i, m_j) = -\frac{1}{2} m_i^2 + \alpha m_i m_j.$$

The unique best-response of agent i is simply the linear function

$$m_i^*(m_j) = \alpha m_j.$$

If $\alpha \geq 0$ then this game is clearly supermodular. If $\alpha \in (0, 1)$ then the game also has a unique Nash equilibrium at $m_0^* = (0, 0)$ and all adaptive dynamics converge to m^*.

If $\alpha > 1$, however, then the game is still supermodular but now it has three Nash equilibria: $\underline{m}^* = (-100, 100)$, $m_0^* = (0, 0)$, and $\overline{m}^* = (100, 100)$. Furthermore, the interior equilibrium is unstable under most adaptive dynamics; a simple Cournot best-response process initiated away from m_0^* will converge monotonically to either $\underline{m}^*$ or $\overline{m}^*$. This is consistent with the Milgrom & Roberts stability result because the interval formed by the smallest and largest Nash equilibria is the entire strategy space, and so the stability result is vacuous for this game.

Now take the same game but extend each $\mathcal{M}_i$ to equal the entire real line. This game now has a unique Nash equilibrium at m_0^* for all $\alpha \neq 1$ but is no longer supermodular for *any* α because the strategy space is not compact. If $\alpha \in (0, 1)$ then the Cournot process will still converge to the unique equilibrium from any initial point, and so the loss of supermodularity is irrelevant for this particular dynamic. If $\alpha > 1$, however, then m_0^* is not even locally stable under the Cournot process and generically the process will move monotonically away from the equilibrium point.

This example shows how requiring supermodularity of all games induced by a mechanism can lead to serious problems. Existing mechanisms–including Chen's 2002 'supermodular' mechanism—use unbounded strategy spaces; if the mechanism's strategy space is not made compact then supermodularity alone cannot guarantee that simple adaptive dynamics like the Cournot process will not diverge rapidly away from equilibrium. If the mechanism's strategy space is arbitrarily capped then this might create new Nash equilibria for certain environments whose outcomes surely will not coincide with any Walrasian or Lindahl equilibrium.

Finally, we show that a mechanism exists that implements Lindahl allocations and is dynamically unstable even though it satisfies all of the requirements for supermodularity except the compact strategy space.

Example 1. Let $\mathcal{M}_i = \mathbb{R}^1$ for each i and suppose the total number of agents (n) is even. Take

[1] In Bayesian environments, Mathevet (2007) considers direct supermodular mechanisms on compact type spaces, so the third requirement is applied.

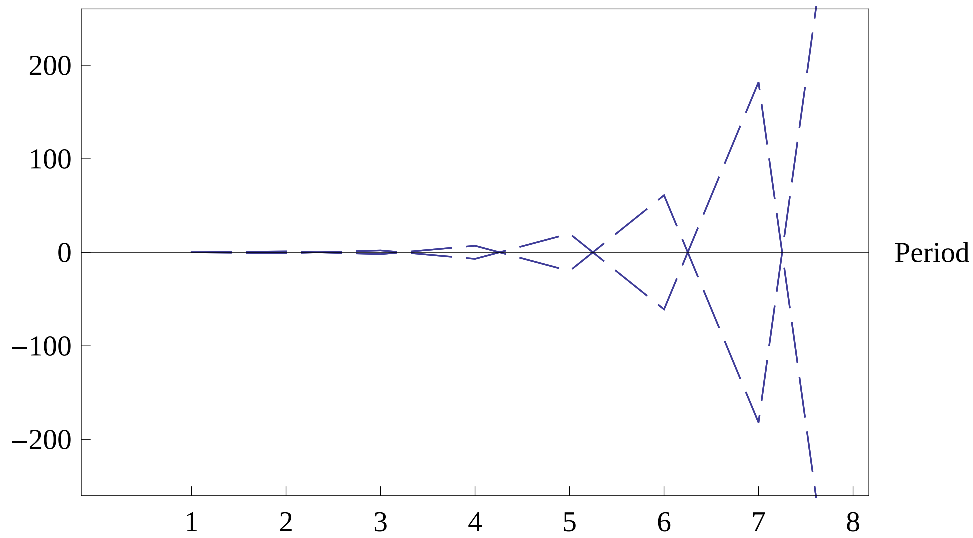

Figure 1: Simulated best-response dynamics in an unstable 'supermodular' game.

any quasilinear public goods environment of the form $v_i(y|\theta) + x_i$ where $v_i'' \in (-B, -1/B)$ for some $B > 1$ and let the constant marginal cost of the public good be $\kappa > 0$. Consider the mechanism given by

$$y(m) = \sum_{i=1}^{n/2} m_i - \sum_{i=n/2+1}^{n} m_i \tag{3}$$

and

$$q_i(m) = \begin{cases} \frac{\kappa}{n} - \gamma \sum_{j \neq \{i, i+\frac{n}{2}\}} m_j & \text{if} \quad i \leq n/2 \\ \frac{\kappa}{n} + \gamma \sum_{j \neq \{i, i+\frac{n}{2}\}} m_j & \text{if} \quad i > n/2. \end{cases} \tag{4}$$

It is easily verified that if $\gamma > B$ then $\partial^2 U_i(m_i, m_{-i}|\theta_i)/\partial m_i \partial m_j > 0$ for all i and $j \neq i$. One can show that this mechanism also implements the Lindahl correspondence using a proof very similar to that of Walker (1981). Calculating the total derivatives of the individual best-response functions $\beta_i(m_{-i})$, however, gives

$$\sum_{j \neq i} \frac{\partial \beta_j(m_{-j})}{\partial m_i} = 1 + (n-2) \frac{\gamma}{-v_j''(y|\theta_j)},$$

which is at least $n - 1$ since $\gamma > \max_j v_j''$. Thus, a one-unit change in m_i leads to a total shift of at least $(n-1)$ units in m_{-i}. Figure 1 shows a simulation of the best-response dynamics for the case of $n = 4$, $v_i(y|\theta_i) = -(1/2)(y-2)^2$ for each i, $\kappa = 4$, and mechanism parameter $\gamma = 2$. Clearly, $\{m(t)\}$ diverges exponentially and is unstable.

3.2 Contractive Games and Learning

Given the above difficulties, we follow a different tack, suggested first by Van Essen (2009). In the example above, what distinguishes stability from instability—at least for the Cournot process—is whether or not $\alpha < 1$. This coincides with whether or not the best-response functions form a contraction mapping in $\mathcal{M}$. In general, if a game's best-response function is a contraction mapping, then its equilibrium will be unique. We also show below that a wide range of learning dynamics will also converge to the unique equilibrium from any starting point. Therefore, in our quest for stability, we focus on guaranteeing that a mechanism induces games whose best-response functions define contraction mappings.

Definition 1. If $(\mathcal{M}, d)$ is a metric space then a (single-valued) function $\beta : \mathcal{M} \to \mathcal{M}$ is a *d-contraction mapping* if there is some constant $\xi \in (0, 1)$ such that for all $m, m' \in \mathcal{M}$,

$$d(\beta(m), \beta(m')) \leq \xi \, d(m, m').$$

When the metric d is understood we simply refer to β as a *contraction mapping*.

Definition 2. Let $(\mathcal{M}, d)$ be a complete metric space. A mechanism $\Gamma = (\mathcal{M}, h)$ with outcome function $h : \mathcal{M} \to \mathcal{Z}$ is *contractive on* Θ if for every $\theta \in \Theta$ the induced game $(\mathcal{I}, \mathcal{M}, (u_i(h(\cdot)|\theta_i))_i)$ has a single-valued best-response function $\beta : \mathcal{M} \to \mathcal{M}$ that is a contraction mapping.

Contractiveness is a strong property to require of a mechanism; by the Banach fixed point theorem it guarantees the existence of a single-valued mapping $\mu : \Theta \to \mathcal{M}$ such that $\mu(\theta)$ is the unique Nash equilibrium of the game induced by Γ for each $\theta \in \Theta$. As the proof of Banach's theorem illustrates, the Cournot best-response dynamic is globally stable and converges to $\mu(\theta)$ for each θ. If this myopic dynamic process is an accurate description of how humans play repeated instances of a mechanism then the contraction property guarantees convergence to the equilibrium point; if the mechanism Γ also implements some social choice function f in Nash equilibrium then in the limit agents will converge to outcome realizations in $f(\theta)$ regardless of θ.

The experimental literature on learning dynamics suggests that the adaptive processes that best describe human behavior are more complex and subtle than the simple Cournot best-response dynamic. Milgrom and Roberts (1990) show that the family of adaptive dynamics that are globally stable under a supermodular game is quite large; almost all well-known dynamics are included in the family they describe. We provide an analogue of this result for contractive games: there is a family of adaptive dynamics that contains the family described by Milgrom and Roberts (1990) such that every dynamic in this family is globally stable in any game with a contractive best response function. In other words, we find a larger set of dynamics that are globally stable under contractive games than those known to be stable under supermodular games; therefore, there is no loss in the scope of the stability results when moving from supermodular games to contractive games.

To discuss dynamics formally, consider a game (possibly induced by a mechanism at some type profile θ) with a contractive best-response function $\beta : \mathcal{M} \to \mathcal{M}$ where $(\mathcal{M}, d)$ is a complete metric space and $m^* \in \mathcal{M}$ is the unique fixed point of β. Following Milgrom and Roberts (1990), denote any dynamic process by $\{m(t)\}$ with $m(t) \in \mathcal{M}$ where the 'time' t belongs to some linearly ordered index set $\mathcal{T}$ that may be finite or infinite. Let $H(t', t) = \{m(s) : t' \leq s < t\}$ denote the history of play from time t' up to (but not including) t. For any $r \geq 0$ let $B_d(r|m^*) = \{m \in \mathcal{M} : d(m, m^*) \leq r\}$ be the closed ball with center m^* and radius r. For any bounded set $\mathcal{M}' \subset \mathcal{M}$ define

$$B_d(\mathcal{M}') = \bigcap \{B_d(r|m^*) : \mathcal{M}' \subseteq B_d(r|m^*)\}$$

to be the smallest closed (and, therefore, compact) ball centered at m^* that includes $\mathcal{M}'$. When d is understood we drop the subscript. Since the image of a compact set under a contraction correspondence is compact, we can give the following definition.

Definition 3. A learning dynamic $\{m(t)\}$ is an *adaptive best-response dynamic* (*ABR dynamic*) if $(\forall t' \in \mathcal{T})(\exists \hat{t} > t')(\forall t \geq \hat{t})$, $m(t) \in B(\beta(B(H(t', t))))$.

To understand this definition, recall that $B(H(t', t))$ is the smallest compact ball centered at m^* that contains the history of play from t' up to (but not including) t. $B(\beta(B(H(t', t))))$ is the smallest compact ball centered at m^* that contains all best responses to $B(H(t', t))$. The requirement that $m(t) \in B(\beta(B(H(t', t))))$ means that strategies at date t are either a best response to some point in $B(H(t', t))$ or are at least in the compact ball centered at m^* that contains $B(H(t', t))$. Thus, $m(t)$ takes $H(t', t)$, forms a 'belief' that opponents will play some strategy in $B(H(t', t))$, and chooses any strategy that is either a best response to this belief or is at lest no farther from equilibrium than any best response to this belief. The quantifiers then say that for any date t' there is some later date $\hat{t}$ after which each player only considers the history of play from t' to the current point in time; in other words, after $\hat{t}$ the dynamic ignores the history of play prior to t'.

The family of ABR dynamics allows for pure best response play, dynamics that best respond to some weighted average of past play, dynamics with bounded memory, and even dynamics that put some weight on the equilibrium of the game. This class of dynamics is quite large, indeed; if d is the ℓ_∞-norm then for any

There is a wide class of learning dynamics that converge to the equilibrium in contractive games. These dynamics look at *everything* in a radius including past play, and allow for 'averaging' among any such profiles. This definition encompasses Cournot dynamics, and many processes with bounded memory.

Theorem 1. If a game generates a contractive best-response function then any adaptive best-response dynamic converges to a unique Nash equilibrium.

4 Necessary Conditions for Nash Implementation

Our ultimate goal is to describe a procedure for designing mechanisms that Nash implement Walrasian or Lindahl equilibria and have desirable stability properties. To do this we first identify what all mechanisms that Nash implement Walrasian or Lindahl equilibria must look like. Then, given this necessary condition on the form of the mechanism, we can show how to modify any such mechanism to guarantee the required stability properties. In this section we focus on the necessary condition; stability is covered in the following section.

4.1 One-Dimensional Mechanisms

In this subsection we restrict attention to mechanisms whose strategy spaces are one-dimensional, meaning that $\mathcal{M}_i = \mathbb{R}^1$ for each $i \in \mathcal{I}$. It is well-known that such mechanisms

Our proofs rely heavily on differentiability arguments, so we restrict attention to those mechanisms whose messages whose outcome functions are twice continuously differentiable in every agent's message.

Assumption 1 (Differentiability)**.** For each agent i the message space $\mathcal{M}_i$ equals $\mathbb{R}^1$ and, for each message vector $m \in \mathcal{M}$ the functions x_i and y_i are twice continuously differentiable in m_i at m.

Consider now an agent i who believes the other agents will submit a message vector m_{-i}. Given a twice continuously differentiable mechanism $(y_i, , q_i, g_i)_i$, agent i is able to trace out a continuously differentiable manifold of (x_i, y_i) pairs that he can unilaterally achieve by varying m_i, holding fixed m_{-i}. Identifying an agent's best-response message m_i^* in response to m_{-i} is then equivalent to identifying the point on this manifold that maximizes i's utility.

Our next assumption explicitly rules out cases where agent i's manifold becomes arbitrarily flat in (x_i, y_i)-space by requiring $\partial y_i/\partial m_i$ to be uniformly bounded away from zero. To the authors' knowledge this does not rule out any existing mechanisms; most use linear function such as $y_i(m) = \sum_j m_j$.

Assumption 2 (Responsive y_i)**.** For each i there exists some $\varepsilon_i > 0$ such that for all $m \in \mathcal{M}$, $|\partial y_i(m)/\partial m_i| \geq \varepsilon$.

Under assumption 2' each agent i's manifold can now be thought of as the graph of a single-valued mapping from y_i into x_i. We denote this mapping by

$$\chi_i(y_i|m_{-i}) := x_i(y_i^{-1}(y_i|m_{-i}), m_{-i}),$$

where $y_i^{-1}(y_i|m_{-i})$ identifies the unique m_i such that $y_i(m_i, m_{-i}) = y_i$. We let y_i be the input into this function to highlight the fact that agent i, through his choice of m_i can choose *any* level of y_i since the function $y_i(\cdot, m_{-i})$ is bijective for all m_{-i} by assumption 2'.

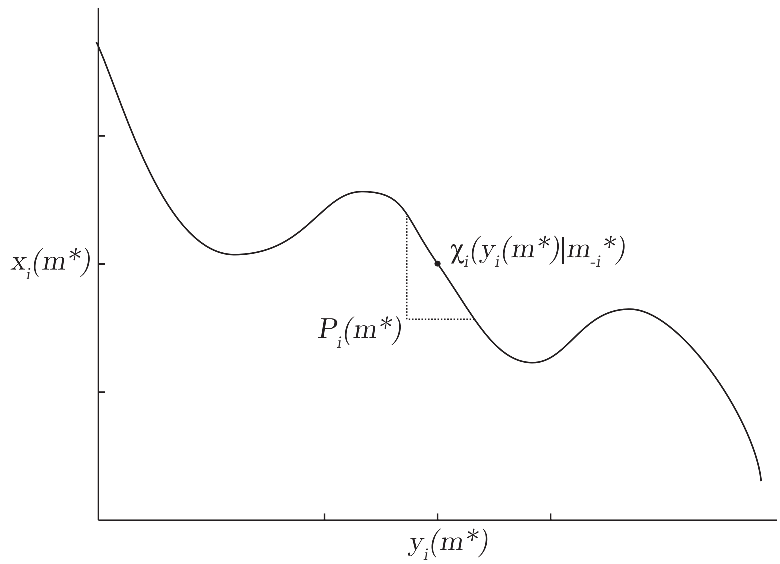

Figure 2: The mapping $\chi_i(y_i|m_{-i})$ and the effective price $P_i(m)$ at m^*.

We show an example of $\chi_i(y_i|m_{-i})$ in figure 2. At the point $m = m^*$ the outcome $(x_i(m^*), y_i(m^*))$ is realized by agent i. As i differentially changes his message m_i, he differentially changes his allocation (x_i, y_i) along the graph of χ_i. In this example, if $\partial y_i/\partial m_i > 0$ then an increase in m_i leads to an increase in y_i but a decrease in x_i; thus, the agent is effectively trading off units of these two goods at a rate determined by the downward slope of $\chi_i(y_i(m^*)|m^*_{-i})$. This downward slope—which we label $P_i(m^*)$—serves as the *effective price* of y_i charged by the mechanism at m^*. Specifically, the effective price of y_i at m is given by

$$P_i(m) = -\frac{\partial x_i(m)/\partial m_i}{\partial y_i(m)/\partial m_i}. \tag{5}$$

The effective price in a mechanism serves the same role *locally* as prices in a Walrasian or Lindahl equilibrium. To see this, write agent i's utility at θ_i in the game induced by some mechanism $(\mathcal{M}_i, x_i, y_i)_{i\in\mathcal{I}}$ as

$$U_i(m|\theta_i) := u_i(x_i(m), y_i(m)|\theta_i). \tag{6}$$

Since the message space is open, at any Nash equilibrium point m^* it must be that the first order condition

$$\frac{\partial u_i(x_i(m^*), y_i(m^*)|\theta_i)}{\partial x_i}\frac{\partial x_i(m^*)}{\partial m_i} = -\frac{\partial u_i(x_i(m^*), y_i(m^*)|\theta_i)}{\partial y_i}\frac{\partial y_i(m^*)}{\partial m_i}$$

is satisfied; but this is equivalent to

$$\frac{\partial u_i(x_i, y_i|\theta_i)/\partial y_i}{\partial u_i(x_i, y_i|\theta_i)/\partial x_i} = P_i(m^*), \tag{7}$$

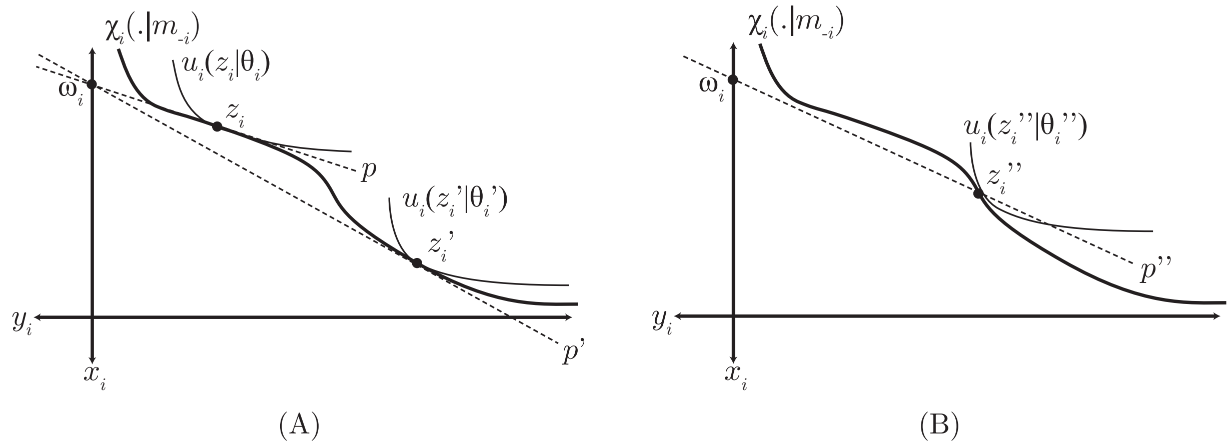

Figure 3: (A) The Triple Tangency Property, and (B) a 'bad' Nash equilibrium.

so that the marginal rate of substitution between y_i and x_i equals the effective price of the mechanism at m^*. If this mechanism implements a Walrasian or Lindahl equilibrium, then the ratio of marginal utilities must also equal the Walrasian or Lindahl price. Thus, the effective prices at the equilibrium message profile m^* must match the Walrasian or Lindahl price for each environment θ. This leads to the following observation:

Observation (The Triple Tangency Property). At any Nash equilibrium point each agent's indifference curve in (x_i, y_i)-space must be tangent to both the mechanism's outcome manifold $\chi_i(\cdot|m^*_{-i})$ and the Walrasian or Lindahl equilibrium's price hyperplane.

The Triple Tangency Property is illustrated in panel (A) of figure 3; for type θ_i the point z_i is both a Nash equilibrium outcome and a Walrasian allocation at equilibrium price p. Similarly, z'_i is a Nash equilibrium point and a Walrasian allocation (at price p') for type θ'_i.

Now consider panel (B) if figure 3. If the type space is sufficiently 'rich'—meaning that every outcome z is a Nash equilibrium outcome for some environment—then there will exist some $\theta'' \in \Theta$ such that the point z''_i is also a Nash equilibrium outcome; however, this must be a 'bad' equilibrium because it cannot possibly be a Walrasian equilibrium for this environment. This mechanism therefore cannot fully implement the Walrasian correspondence when θ'' is an admissible type profile.

If the type space is rich then *every* point z_i along χ_i can be made into a Nash equilibrium outcome by selecting an appropriate type profile. If this is the case then any mechanism that Nash implements the Walrasian or Lindahl correspondence must have a linear χ_i function for each i that extends through ω_i; if it does not then a 'bad' equilibrium outcome such as z''_i can be found. We now make this richness assumption explicit.

Assumption 3 (Rich Type Space)**.** $\nu(\Theta) = \mathcal{M}$.

The exact restrictions assumption 3 places on the set Θ depends crucially on the shape of the mechanism (or, conversely, the assumed size of Θ crucially affects the assumptions made on the shape of the mechanism). To better understand assumption 3, we proceed by identifying two separate assumptions on Θ and on the mechanism that together imply assumption 3.

Assumption 3a (Quadratic Utilities)**.** For each vector $(\alpha_i, \beta_i)_{i=1}^n \in (\mathbb{R}_{++} \times \mathbb{R})^n$ there is some $\theta \in \Theta$ such that for each i,

$$u_i(x_i, y_i|\theta_i) = \left(-\alpha_i y_i^2 + \beta_i y_i\right) + x_i.$$

Assumption 3b (Weak Hölder Continuity)**.** For all $m \in \mathcal{M}$ and $i \in \mathcal{I}$ there exists some finite $\gamma_i(m) > 0$ such that for all m_i',

$$|x_i(m_i', m_{-i}) - x_i(m)| \leq \gamma_i(m) \max\left\{|y_i(m_i', m_{-i}) - y_i(m)|^2, |y_i(m_i', m_{-i}) - y_i(m)|^{1/2}\right\}$$

To interpret Assumption 3b, first consider changes in m_i that lead to large changes in y_i. In this case the squared term in the maximand applies, and so the assumption places quadratic upper and lower bounds on the change in x_i. For changes in m_i that lead to small changes in y_i the upper and lower bounds are square-root bounds. In either case the requirement is strictly weaker than requiring that χ_i be Hölder continuous of degree 2 or that χ_i be Lipschitz continuous. The bounds on χ_i imposed by this assumption are demonstrated in figure 4.

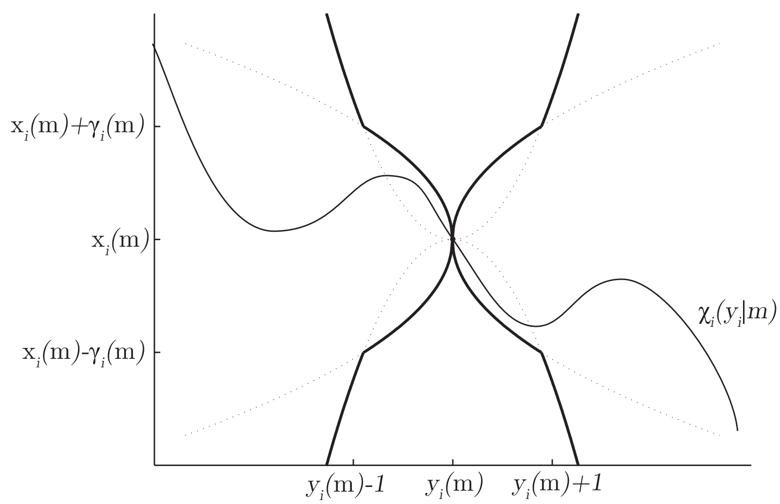

Figure 4: The bounds on $\chi_i(y_i|m_{-i})$ imposed by assumption 3b.

Under assumption 3a we can define Θ_2 as the subset of Θ containing all preferences of this form; formally, let

$$\Theta_2 := \left\{\theta \in \Theta : (\forall i \in \mathcal{I})\ u_i(x_i, y_i|\theta_i) = \left(-\alpha_i y_i^2 + \beta_i y_i\right) + x_i\right\}. \tag{8}$$

We now verify that these two assumptions imply assumption 3.

Proposition 1. Assumptions 1, 2, 3a and 3b imply assumption 3.

The above argument sketches the proof of our first result:

Theorem 2. Under assumptions 1, 2, and 3, if a mechanism Γ Nash implements the Walrasian or Lindahl correspondence with $\mathcal{M}_i = \mathbb{R}^1$ for each i then for every $i \in \mathcal{I}$ and *every* $m \in \mathcal{M}$,

$$x_i(m) \equiv -q_i(m_{-i})y_i(m) \tag{9}$$

so that q_i does not depend on m_i and $g_i(m) \equiv 0$.

Theorem 2 gives a strong but intuitive result: If a mechanism is to Nash implement the Walrasian or Lindahl correspondence, then each agent's message-choosing problem in the mechanism (taking others' messages as fixed) must be identical to the quantity-choosing problem in an exchange economy; in both cases agents select an optimal quantity of y_i (either by choosing m_i in the mechanism or by choosing y_i directly) taking as given the per-unit price q_i. Conversely, if an agent has the ability to change both his chosen quantity *and* his per-unit price then such a mechanism cannot always implement Walrasian or Lindahl allocations.

For the case of public goods economies, compare theorem 2 with the mechanisms of Groves and Ledyard (1977), Walker (1981), and Tian (1990). All three are one-dimensional mechanisms, but the Groves-Ledyard mechanism has a non-trivial penalty function while the latter two do not. Consequently, Walker's and Tian's mechanisms Nash implement the Lindahl correspondence while the Groves-Ledyard only selects Pareto optimal public goods levels but fails to charge Lindahl prices, leading to violations of the individual rationality constraint (see Hurwicz, 1979).

For private goods economies theorem 2 is in fact excessively strong. If no agent is allowed to affect their own per-unit price, if all agents must have the same price at every equilibrium message, and if every message is an equilibrium message for some type profile, then the only admissible price function q_i is a constant function that depends on *no* agents' reports. But clearly such a mechanism cannot fully implement the Walrasian correspondence on a rich type space, and so we arrive at a contradiction: There cannot exist a one-dimensional mechanism that Nash implements the Walrasian correspondence. This result is not new; Reichelstein and Reiter (1988) develop lower bounds on the dimensionality of the message space that preclude one-dimensional mechanisms. They also recognize that any mechanism that Nash implements the Walrasian correspondence must induce 'price-taking' behavior, but this is infeasible with only one dimension per agent.

Corollary 1 (See also Reichelstein and Reiter, 1988)**.** Under assumptions 1–3 there does not exist a one-dimensional mechanism that Nash implements the Walrasian correspondence.

4.2 Higher-Dimensional Mechanisms

When the strategy space of each agent has multiple dimensions, we can further break apart a mechanism by distinguishing those strategy space dimensions that affect the quantity of the non-numeraire good from those that do not. Specifically, we let $\mathcal{M}_i = \mathcal{R}_i \times \mathcal{S}_i$, where, for each i, $\mathcal{R}_i \subseteq \mathbb{R}^{J_i}$ represents those dimensions $k \in \{1, \ldots, J_i\}$ for which there is some message vector $m = (r, s)$ such that $\partial y_i(r, s)/\partial r_{ik} \neq 0$. The set $\mathcal{S}_i \subseteq \mathbb{R}^{K_i - J_i}$ represents those dimensions $k \in \{1, \ldots, K_i - J_i\}$ for which $\partial y_i/\partial s_{ik} \equiv 0$.

With the partitioning of the strategy spaces into $\mathcal{R}_i$ and $\mathcal{S}_i$ we can modify equation 2 slightly and write any mechanism's numéraire outcome function as

$$x_i(r, s) = -q_i(r, s) y_i(r) - g_i(r, s). \tag{10}$$

Unlike equation 2, this formulation allows the 'price' term q_i to depend on agent i's message. We will show, however, that along the equilibrium set of announcements agent i cannot affect q_i, even though he may be able to affect q_i at points off the equilibrium set.

We now reformulate our previous assumptions for the case of multiple dimensions.

Assumption 1' (Differentiability)**.** For each agent i and each message vector $m \in \mathcal{M}$ the functions x_i and y_i are twice continuously differentiable in every dimension at m.

Assumption 2' (Responsive y_i)**.** For each i there exists some $\varepsilon_i > 0$ such that for all $r \in \mathcal{R}$ and all dimensions $k \in \{1, \ldots, J_i\}$, $|\partial y_i(r)/\partial r_{ik}| \geq \varepsilon$.

As for assumption 3 with extra dimensions, it becomes overly restrictive to assume that $\nu(\theta) = \mathcal{M}$ because the dimensionality of $\nu(\Theta)$ may be strictly less than that of $\mathcal{M}$. As a simple example, take any mechanism with one-dimensional message spaces in which agents send $m_i \in \mathbb{R}^1$ and the outcome is given by $(\tilde{x}_i(m), \tilde{y}_i(m))$ for each i. Assume that this mechanism implements some social choice correspondence $f(\theta)$. Now consider a new mechanism in which agents send two-dimensional messages of the form $m_i = (r_i, s_i)$ and the outcomes are given by $\tilde{y}_i(r)$ and $\tilde{x}_i(r) - |s_i|$. Clearly, if m^* is a Nash equilibrium of the original mechanism at type profile θ then $(m^*, 0)$ is a Nash equilibrium of the new mechanism at θ and so this new mechanism also implements f; however, the equilibrium set $\nu(\Theta)$ is restricted to lie in $\mathcal{R} \times \{0\}^n$ and thus cannot equal $\mathcal{M}$.

More generally, take any mechanism $(x_i(r, s), y_i(r))_i$ and note that because s only enters into the determination of the numeraire, and utility is strictly increasing in the numeraire, then the mechanism designer can predict those values of s that will be chosen in response to each r without any further information about the shape of agents' preferences. This is done by noting that each agent i must choose

$$s_i \in \sigma_i(r, s_{-i}) := \arg\max_{s_i' \in \mathcal{S}_i} \; x_i(r, s_i', s_{-i})$$

and so—ruling out mixed strategies—the mechanism designer can predict that in equilibrium agents will select

$$s \in \sigma(r) := \{s^* \in \mathcal{S} : (\forall i \in \mathcal{I}) s_i^* \in \sigma_i(r, s_{-i}^*)\}.$$

If $\sigma(r')$ is empty for some r' then there cannot be an equilibrium of the form (r', s) for any $s \in \mathcal{S}$.

The above argument shows that $\nu(\Theta) \subseteq \{(r,s) \in \mathcal{R} \times \mathcal{S} : s \in \sigma(r)\}$. There is one additional restriction on $\nu(\Theta)$ that can be made without knowing agents' preferences: For each agent i and dimension $k \in \{1, \ldots, J_i\}$ define the effective price along dimension k at message (r,s) by

$$P_{ik}(r,s) := -\frac{\partial x_i(r,s)/\partial r_{ik}}{\partial y_i(r)/\partial r_{ik}}$$

and note that by the same first-order argument as in the one-dimensional case, it must be that $P_{ik}(r,s)$ equals i's marginal rate of substitution between y_i and x_i at any equilibrium point (r,s). Therefore, if a point (r',s') is such that $P_{ik}(r',s') \neq P_{il}(r',s')$ then $(r',s') \notin \nu(\Theta)$. Intuitively, this would represent a situation where agent i could, by changing different dimensions of r_i, trade off x_i for y_i at two different effective prices. But then it cannot be that (r',s') is a local maximizer of a smooth utility function since it cannot be that both P_{ik} and P_{il} define supporting hyperplanes for i's convex upper contour set at $(x_i(r',s'), y_i(r',s'))$.

Given these two restrictions, we now define

$$\begin{aligned} \mathcal{M}^* \quad := \quad & \{m \in \mathcal{M} : (\forall i \in \mathcal{I})(\forall k, l \in \{1, \ldots, J_i\}) \ P_{ik}(m) = P_{il}(m)\} \\ & \cap \{m = (r,s) \in \mathcal{M} : s \in \sigma(r)\} \end{aligned} \tag{11}$$

to be the set of 'candidate equilibrium' points in $\mathcal{M}$. The natural extension of our rich type space assumption (assumption 3) is that every candidate equilibrium is in fact an equilibrium for some environment.

Assumption 3'. $\nu(\Theta) = \mathcal{M}^*$.

As in the one-dimensional case, assumption 3' places joint restrictions on the type space and the mechanism. To separate these restrictions we can replace assumption 3' with assumption 3a (Quadratic Utilities, given above) and

Assumption 3'b (Weak Hölder Continuity)**.** For all $r \in \mathcal{R}$, all $s \in \sigma(r)$, and all $i \in \mathcal{I}$ there exists some finite $\gamma_i(r) > 0$ such that for all $r_i' \in \mathcal{R}_i$ and $s_i' \in \sigma_i(r_i', r_{-i}, s_{-i})$,

$$|x_i(r_i', r_{-i}, s_i', s_{-i}) - x_i(r,s)| \leq \gamma_i(r) \max\left\{|y_i(r_i', r_{-i}) - y_i(r)|^2, |y_i(r_i', r_{-i}) - y_i(r)|^{1/2}\right\}$$

Sufficiency of these two assumptions is then given by the following proposition.

Proposition 1'. Assumptions 1', 2', 3a, and 3'b imply assumption 3'.

Finally, we provide a definition of *regular* candidate equilibrium points for which our theorem will apply.

Definition 4. A candidate equilibrium $(r^*, s^*) \in \mathcal{M}^*$ is *regular* if for each i there is some open set $\mathcal{R}_i^0$ containing r_i^* and a differentiable function $\varsigma_i : \mathcal{R} \times \mathcal{S}_{-i} \to \mathcal{S}$ such that $\varsigma_i(r^*, s_{-i}^*) = s_i^*$ and $(r_i', r_{-i}^*, \varsigma_i(r_i', s_{-i}^*), s_{-i}^*) \in \mathcal{M}^*$ for all $r_i' \in \mathcal{R}_i^0$.

A candidate equilibrium (r^*, s^*) is regular if differential deviations in r_i are accompanied by a differential change in s_i and the deviation does not lead to a strategy profile outside of the candidate equilibrium set. An example of a non-regular equilibrium would be one for which differential changes in r_i lead to points at which $\sigma(r)$ is empty, or at which $P_{ik} \neq P_{il}$ for some dimensions k and l. We refer to ς_i as i's adjustment function.

We now prove a higher-dimensional analog to theorem 2.

Theorem 3. Suppose a mechanism $\Gamma = (\mathcal{M}_i, x_i, y_i)_{i \in \mathcal{I}}$ Nash implements the Lindahl or Walrasian correspondences and satisfies assumptions 1', 2', and 3'. Writing the mechanism as

$$x_i(r, s) = -q_i(r, s) y_i(r) - g_i(r, s),$$

it must be the case that for every regular point $(r^*, s^*) \in \mathcal{M}^*$ with adjustment functions $(\varsigma_i)_i$,

$$\frac{dq_i(r^*, \varsigma_i(r^*, s_{-i}^*), s_{-i}^*)}{dr_{ik}} = 0 \quad \forall i \in \mathcal{I}, k \in \{1, \ldots, J_i\}$$

and

$$g_i(r^*, s^*) = 0.$$

Thus, higher dimensional mechanisms may allow agents to affect their own prices and face non-trivial penalty functions, but the penalty function must equal zero on the equilibrium set and each agent's price must not change as the agent unilaterally changes r_i and adjusts s_i appropriately. At off-equilibrium or non-regular equilibrium points, however, we derive no restrictions on the shape of the mechanism. It is this freedom that allows us to introduce global stability properties into a mechanism. Intuitively, one should be able to take a mechanism satisfying the restrictions of theorem 3 and adjust the mechanism on $\mathcal{M} \backslash \mathcal{M}^*$ so that any adaptive dynamic process that wanders off of $\mathcal{M}^*$ will eventually return back to the appropriate point in $\mathcal{M}^*$, restoring the equilibrium.

5 Contractive Mechanisms

From this point forward we make the following assumption:

Assumption 4. For all types $\theta \in \Theta$ all agents i have quasilinear preferences of the form $v_i(y_i|\theta_i)+x_i$ where $v_i' > 0$ and there is some $B > 0$ such that $v_i'' \in (-B, -1/B)$.

We now show that there cannot exist a mechanism with one-dimensional strategy spaces ($\mathcal{M}_i = \mathbb{R}^1$ for each i) that Nash implements the Lindahl or Walrasian correspondence under our maintained assumptions.

Theorem 4. Under assumptions 1–4 there does not exist a mechanism with $\mathcal{M}_i = \mathbb{R}^1$ for each i that is both contractive and Nash implements the Lindahl or Walrasian correspondence.

Note that proof for the Walrasian correspondence is trivial since there does not exist *any* one-dimensional mechanism that implements the Walrasian allocations.

Inspection of the proof reveals that Theorem 4 holds true even if v_i'' can take any value in $(-\infty, 0)$; the bounds on $v_i i''$ from Assumption 4 are needed in the sequel to generate higher-dimensional mechanisms that are contractive. One can show that if v_i'' is allowed to be arbitrarily negative or arbitrarily close to zero then no mechanism that Nash implements the Lindahl or Walrasian correspondences can be contractive. The intuition is straightforward: If v_i'' is extremely negative then preferences for the public good dominate the incentive effects created by the mechanism and the resulting public goods provision game becomes dynamically unstable. If v_i'' is arbitrarily close to zero then small changes in r_j that affect q_i lead to arbitrarily large shifts in i's demand for y_i (because linear preferences have discontinuous demand functions), causing arbitrarily large shifts in the optimal choice of r_i.

5.1 The Lindahl Correspondence

This above impossibility result for single-dimensional mechanisms forces us to consider higher-dimensional mechanisms when seeking contractive implementation of the Lindahl correspondence. We now present a two-dimensional mechanism that Nash implements the Lindahl correspondence and is contractive on Θ.

Let $\mathcal{M}_i = \mathcal{R}_i \times \mathcal{S}_i$ for each i with $\mathcal{R}_i = \mathcal{S}_i = \mathbb{R}^1$, choose $\delta > 0$, and set

$$y(r) = \sum_i r_i, \tag{12}$$

$$q_i(m_{-i}, s_{-i}) = \left(\frac{\kappa}{n} + r_{i-1} - r_{i+1}\right) + \delta\frac{n-1}{n^2}(s_{i-1} - \frac{1}{n}r_{i+1}), \tag{13}$$

and

$$g_i(r, s) = \frac{1}{2}(s_i - \frac{1}{n}r_{i+1})^2 + \frac{\delta}{2}(s_{i-1} - \frac{1}{n}r_i)^2. \tag{14}$$

Theorem 5. For large δ the mechanism defined by equations 12–14 fully Nash implements the Lindahl correspondence and is contractive on Θ.

5.2 The Walrasian correspondence

At this time it is not known whether or not there exists a contractive mechanism that Nash implements the Walrasian correspondence; instead we present a two-dimensional contractive mechanism that Nash implements the ε-Walrasian correspondence.

Let $\mathcal{M}_i = \mathcal{R}_i \times \mathcal{S}_i$ for each i with $\mathcal{R}_i = \mathcal{S}_i = \mathbb{R}^1$ and set

$$y_i(r, s_{-i}) = (r_{i-1} - r_{i+1}) - \frac{\delta}{n}\left(s_{i+1} - \frac{n+1}{n} r_i\right) \tag{15}$$

$$q_i(s_{-i}) = \frac{1}{n-1}\sum_{j\neq i} s_j \tag{16}$$

$$g_i(r, s) = (s_i - \delta\frac{n+1}{n^2}\sum_j r_j)^2 \tag{17}$$

Since $\sum_i y_i = 0$, it is balanced in the second good. Out of equilibrium, however, the designer may collect or pay out extra cash.

Theorem 6. The mechanism defined by equations 15–17 fully Nash implements the ε-Walrasian correspondence and, under assumption 4, there is some $\delta > 0$ such that the mechanism is contractive on Θ.

6 Game-Theoretic Foundation for Walrasian Equilibria

General equilibrium theory has long been criticized for providing poor justifications of the Walrasian equilibrium. There is no plausible theory of how economies attain competitive equilibrium (see Kirman, 1989), and some researchers also question the rationality postulate on the agents (Simon, 1978, e.g.).

Our paper provides mechanisms that help guide boundedly rational agents to play equilibrium profiles whose outcomes are Walrasian allocations. So, we offer a game-theoretic explanation of how competitive equilibrium can emerge using mechanisms that differ from the standard (though vaguely-defined) competitive mechanism. Our explanation uses as its foundations the literature on bounded rationality and learning (Fudenberg and Levine, 1998). While viewing the economy as a non-cooperative game is not new (Shapley and Shubik, 1977; Chatterji and Ghosal, 2004; Gul and Stacchetti, 1999; and Milgrom and Strulovici, 2009), there have been few studies concerned with bounded rationality and convergence in competitive economies (see Crockett et al., 2008, for one exception).

Our contribution stems from the limits of Walrasian tâtonnement, which is the first well-defined dynamics to formalize a market equilibration process. Despite its intuitive appeal—operating through excess demand—it lacks certain foundational components. For example, the forces at work behind the price adjustment and the behavioral assumptions on the agents are unclear. Moreover, many economies may not be able to attain a competitive equilibrium via such a process; although we focus on quasilinear economies where stability and uniqueness are guaranteed when preferences for the non-numéraire good are strictly convex and monotonic (see Brown and Calsamiglia, 2007 or Hildenbrand, 1983), we do not require monotonicity and so the known stability results do not apply. Without monotonicity, stability—even existence—is not guaranteed; e.g., goods may not be gross substitutes even though there are no income effects. Furthermore, equilibrium uniqueness only means that there is a unique price ratio for which excess demands are null; this price ratio could correspond to several Walrasian allocations. A dynamic adjustment process can help in this situation to determine to which allocations consumers will converge.

A Proofs

Theorem 1

Proof. The proof follows by induction. Pick a starting time t_0. By definition of an ABR dynamic, for each point in time t_n there exists some later point in time $t_{n+1} > t_n$ such that for all $t \geq t_{n+1}$, $m(t) \in B(\beta(B(H(t_n, t))))$. For each $n \in \{0, 1, 2, \ldots\}$ let $\mathcal{M}_n = H(t_n, t_{n+1})$ be the history of play from t_n to t_{n+1}.

For any metric d on $\mathcal{M}$, any set $\mathcal{M}' \subseteq \mathcal{M}$, and any point $m' \in \mathcal{M}$ the d-Hausdorff distance between $\mathcal{M}'$ and the singleton set $\{m'\}$ is given by

$$h_d(\mathcal{M}', m') = \sup_{m \in \mathcal{M}'} d(m, m').$$

Therefore, for any set $\mathcal{M}' \subseteq \mathcal{M}$, $h_d(\mathcal{M}', m^*) = h_d(B(\mathcal{M}'), m^*)$. Thus,

$$\begin{aligned}
\xi\, h_d(\mathcal{M}_0, m^*) &= \xi\, h_d(B(\mathcal{M}_0), m^*) \\
&\geq h_d(\beta(B(\mathcal{M}_0)), m^*) \\
&= h_d(B(\beta(B(\mathcal{M}_0))), m^*) \\
&\geq h_d(\mathcal{M}_1, m^*),
\end{aligned}$$

where the first inequality comes from the contraction property of β and the last inequality follows from the fact that $\mathcal{M}_1 \subset B(\beta(B(\mathcal{M}_0)))$. Taking any n and $n+1$, we can use a similar argument to show that $\xi h_d(\mathcal{M}_n, m^*) \geq h_d(\mathcal{M}_{n+1}, m^*)$. Therefore, for all n,

$$\xi^n h_d(\mathcal{M}_0, m^*) \geq h_d(\mathcal{M}_n, m^*),$$

which implies that the sequence $\mathcal{M}_n$ converges to $\{m^*\}$, and so any ABR dynamics converges to m^*. □

Propositions 1 and 1'

Proposition 1 follows from proposition 1', so we only prove the latter. Further, proposition 1' can be weakened to allow for mechanisms that fail the Hölder continuity requirement (assumption 3'b), which would happen if the x_i functions use polynomials of order more than twice as large as the y_i functions. This is done by replacing assumptions 3'b and 3a with the following.

Assumption 5 (Weak Hölder Continuity)**.** Associated with the mechanism Γ is some $\rho \in \{1, 2, \ldots\}$ such that for all $r \in \mathcal{R}$, all $s \in \sigma(r)$, and all $i \in \mathcal{I}$ there exists some finite $\gamma_i(r) > 0$ such that for all $r_i' \in \mathcal{R}_i$ and $s_i' \in \sigma_i(r_i', r_{-i}, s_{-i})$,

$$\left|x_i(r_i', r_{-i}, s_i', s_{-i}) - x_i(r, s)\right| \leq \gamma_i(r) \max\left\{\left|y_i(r_i', r_{-i}) - y_i(r)\right|^{\rho}, \left|y_i(r_i', r_{-i}) - y_i(r)\right|^{1/\rho}\right\},$$

Assumption 6 (Rich Type Space)**.** Let $\hat{\rho} \in \{2, 4, 6, \ldots\}$ be the smallest even value of ρ satisfying assumption 5. For each vector $(\alpha_i, \beta_i)_{i=1}^n \in (\mathbb{R}_{++} \times \mathbb{R})^n$ there is some $\theta \in \Theta$ such that for each i,

$$u_i(x_i, y_i|\theta_i) = \left(-\alpha_i y_i^{\hat{\rho}} + \beta_i y_i\right) + x_i.$$

When $\rho = 2$, assumptions 5 and 6 are identical to assumptions 3'b and 3a, respectively. As ρ increases assumption 5 becomes strictly weaker. Technically, assumption 6 becomes neither stronger or weaker as ρ changes, but in practical terms a higher ρ requires preferences that are more 'exotic' (using higher-order polynomials) and may therefore be viewed as less desirable.

Given these modified assumptions, we can now prove a generalization of proposition 1' for any $\rho \geq 1$; again, the case of $\rho = 2$ reduces to the original statement of proposition $1'$.

Proposition 1'. Take any mechanism $\Gamma = (\mathcal{M}_i, x_i, y_i)_{i \in \mathcal{I}}$ satisfying assumptions 1', 2' and 5 for some $\rho \in \{1, 2, 3, \ldots\}$ and any type space Θ satisfying assumption 6. If $\rho \leq 2$ then every $m \in \mathcal{M}^*$ is a Nash equilibrium of Γ for some $\theta \in \Theta$. If $\rho > 2$ then every $m \in \mathcal{M}^*$ such that $y_i(r) \neq 0$ for all i is a Nash equilibrium of Γ for some $\theta \in \Theta$.

Proof of Proposition 1'. Let $\hat{\rho}$ be the smallest even number weakly greater than ρ. Define $\mathcal{M}^{**}$ by

$$\mathcal{M}^{**} = \left\{(r, s) \in \mathcal{M}^* : (\forall i \in \mathcal{I})\ y_i(r)^{\hat{\rho}-2} \neq 0\right\}.$$

Note that if $\rho \in \{1, 2\}$ then $\hat{\rho} = 2$ and $\mathcal{M}^* = \mathcal{M}^{**}$ (using the convention that $0^0 = 1$). Proposition 1' can then be proven by showing that $\mathcal{M}^{**} \subseteq \nu(\Theta)$. This is done by constructing a mapping $\phi : \mathcal{M}^{**} \to \Theta_{\hat{\rho}}$ (where $\Theta_{\hat{\rho}} \subseteq \Theta$ by assumption 3a) such that $m \in \nu(\phi(m))$ for all $m \in \mathcal{M}^{**}$. Thus,

$$\mathcal{M}^{**} \subseteq \nu(\phi(\mathcal{M}^{**})) = \nu(\Theta_{\hat{\rho}}) \subseteq \nu(\Theta),$$

giving the result.

Specifically, consider the mapping $\phi : \mathcal{M}^{**} \to \Theta_{\hat{\rho}}$ such that $\phi_i(m^*) = (\alpha_i(m^*), \beta_i(m^*)) \in \mathbb{R}_+ \times \mathbb{R}$ for each $m^* \in \mathcal{M}^{**}$ and

$$u_i(x_i, y_i|\phi_i(m^*)) = v_i(y_i|\phi_i(m^*)) + x_i,$$

where

$$v_i(y_i|\phi_i(m^*)) = -\frac{\alpha_i(m^*)}{\hat{\rho}} y_i^{\hat{\rho}} + \beta_i(m^*) y_i$$

and, for a given value of $\alpha_i(m^*)$ (to be determined later in the proof), $\beta_i(m^*)$ is given by

$$\beta_i(r^*, s^*) := \alpha_i(r^*, s^*) y_i^{\hat{\rho}-1}(r) + P_{ik}(r^*, s^*) \tag{18}$$

(recall that P_{ik} is the effective price function defined in equation **??** and does not depend on k since $m^* \in \mathcal{M}^{**}$).

We now fix an arbitrary $m^* = (r^*, s^*) \in \mathcal{M}^{**}$ and show that m_i^* is a best-response to m_{-i}^* for each i in environment $\phi(m^*) = (\alpha_i(m^*), \beta_i(m^*))_{i \in I}$. This is done in two steps; first we verify that m_i^* is a local optimum in response to m_{-i}^* for each i and then we show m_i^* can be made a global optimum by increasing $\alpha_i(m^*)$ sufficiently (allowing $\beta_i(m^*)$ to adjust appropriately as $\alpha_i(m^*)$ changes).

Given $\phi_i(m^*)$, i's objective is to choose (r_i, s_i) to maximize

$$-\frac{\alpha_i(m^*)}{\hat{\rho}} y_i(r_i, r_{-i}^*)^{\hat{\rho}} + \beta_i(m^*) y_i(r_i, r_{-i}^*) + x_i(r_i, s_i, r_{-i}^*, s_{-i}^*). \tag{19}$$

For local optimality, the first-order conditions for each s_{ik} are already satisfied at m^* by the construction of $\mathcal{M}^{**}$ (see equation 11). As for r_{ik}, agent i's first-order condition for utility maximization at (r^*, s^*) with respect to each r_{ik} is

$$\left[-\alpha_i(r^*, s^*) y_i^{\hat{\rho}-1}(r) + \beta_i(r^*, s^*)\right] \frac{\partial y_i(r)}{\partial r_{ik}} + \frac{\partial x_i(r, s)}{\partial r_{ik}} = 0.$$

But the construction of β_i (equation 18) guarantees that this is satisfied at $(r, s) = (r^*, s^*)$ for any $\alpha_i(r^*, s^*)$, so the first-order conditions are satisfied for all $m^* \in \mathcal{M}^{**}$.

To describe the second-order conditions for local optimality, we show that the matrix of second-partial derivatives of i's objective function will be negative definite for sufficiently large $\alpha_i(m^*)$. Shortening notation, let $\mathbf{X_r}$ and $\mathbf{X_s}$ be the column vectors of partial derivatives of x_i with respect to r_i and s_i, respectively, and let $\mathbf{X_{rr}}$, $\mathbf{X_{rs}}$, and $\mathbf{X_{ss}}$ represent the matrices of cross-partial derivatives of x_i. Similarly define $\mathbf{Y_r}$ and $\mathbf{Y_{rr}}$ as the partial and cross-partial derivatives of y_i, respectively. Using this notation, the matrix of second partial derivatives of the objective function (19) (after inserting the definition of $\beta_i(m^*)$ from equation 18) is given by the $K_i \times K_i$ matrix

$$\mathbf{H_i} = \left[\begin{array}{c:c} -\alpha_i(m^*)(\hat{\rho}-1)y_i(r^*)^{\hat{\rho}-2}\left(\mathbf{Y_r}\cdot\mathbf{Y_r}^{\mathrm{T}}\right) + P_{ik}(m^*)\mathbf{Y_{rr}} + \mathbf{X_{rr}} & \mathbf{X_{rs}} \\ \hdashline \mathbf{X_{rs}}^{\mathrm{T}} & \mathbf{X_{ss}} \end{array}\right],$$

where again $P_{ik}(m^*)$ does not depend on k since $m^* \in \mathcal{M}^{**}$. Now take any direction $(\mathbf{d_r}, \mathbf{d_s}) \neq \mathbf{0}$ of deviation from m_i^*. Since $m^* \in \mathcal{M}^{**}$ implies $s^* \in \sigma(r^*)$, we know that any deviation with $\mathbf{d_r} = \mathbf{0}$ will not yield strictly higher utility, hence $(\mathbf{0}, \mathbf{d_s})^{\mathrm{T}} \cdot \mathbf{H_i} \cdot (\mathbf{0}, \mathbf{d_s}) \leq 0$. For any direction $(\mathbf{d_r}, \mathbf{d_s})$ with $\mathbf{d_r} \neq \mathbf{0}$ we have

$$\begin{aligned} (\mathbf{d_r}, \mathbf{d_s})^{\mathrm{T}} \cdot \mathbf{H_i} \cdot (\mathbf{d_r}, \mathbf{d_s}) &= -\alpha_i(m^*)(\hat{\rho}-1)y_i(r^*)^{\hat{\rho}-2}\,\mathbf{d_r}^{\mathrm{T}}\left(\mathbf{Y_r}\cdot\mathbf{Y_r}^{\mathrm{T}}\right)\mathbf{d_r} + K_i(m^*) \\ &= -\alpha_i(m^*)(\hat{\rho}-1)y_i(r^*)^{\hat{\rho}-2}\left(\mathbf{d_r}^{\mathrm{T}}\mathbf{Y_r}\right)^2 + K_i(m^*) \end{aligned}$$

where

$$K_i(m^*) = \mathbf{d_r}^{\mathrm{T}}\left[P_{ik}(m^*)\mathbf{Y_{rr}} + \mathbf{X_{rr}}\right]\mathbf{d_r} + 2\mathbf{d_r}^{\mathrm{T}}\mathbf{X_{rs}}\mathbf{d_s} + \mathbf{d_s}^{\mathrm{T}}\mathbf{X_{ss}}\mathbf{d_s}.$$

Since x_i and y_i are continuously differentiable and $\partial y_i/\partial r_i$ is bounded away from zero, $K_i(m^*)$ is finite for all m^*. Because $y_i(r^*)^{\hat{\rho}-2} \neq 0$, α_i can be chosen to be any function satisfying

$$\alpha_i(m^*) > K_i(m^*)\left((\hat{\rho}-1)y_i(r^*)^{\hat{\rho}-2}\right)^{-1}\left(\mathbf{d_r}^{\mathrm{T}}\mathbf{Y_r}\right)^{-2}$$

for all $m^* \in \mathcal{M}^{**}$, so that $(\mathbf{d_r}, \mathbf{d_s})^{\mathrm{T}} \cdot \mathbf{H_i} \cdot (\mathbf{d_r}, \mathbf{d_s}) < 0$. Thus, m_i^* is a local best-response to m_{-i}^* for large enough $\alpha_i(m^*)$.

We now construct $\phi_i(m^*)$ by increasing $\alpha_i(m^*)$ until m_i^* is a *global* best-response to m_{-i}^*. Since m_i^* is a local best-response, there is some neighborhood $\mathcal{N}_i(m^*)$ of m_i^* on which m_i^* maximizes i's utility given $\alpha_i(m^*)$. Although increasing α_i may change the neighborhood around m^* on which m_i^* is a local best-response, the neighborhood can only increase in size as α_i is increased. Thus, we ignore this dependence of $\mathcal{N}_i(m^*)$ on α_i and show that any $m_i' \notin \mathcal{N}_i(m^*)$ yields a lower payoff than m_i^* when α_i is sufficiently large.

To proceed, pick any m_i' and m_i'' such that $m_i^* \in (m_i', m_i'') \subset \mathcal{N}_i(m^*)$ and, to shorten notation, let $y_i^* = y_i(r^*)$, $x_i^* = x_i(m^*)$, $y_i' = y_i(r_i', r_{-i}^*)$, $x_i' = x_i(m_i', m_{-i}^*)$, $y_i'' = y_i(r_i'', r_{-i}^*)$, and $x_i'' = x_i(m_i'', m_{-i}^*)$.

To show that $u_i(x_i^*, y_i^*) - u_i(x_i', y_i') \geq 0$ for some α_i', we expand this expression to get

$$\alpha_i'\left[\left(\frac{\hat{\rho}-1}{\hat{\rho}}y_i^{*\hat{\rho}} + \frac{1}{\hat{\rho}}y_i'^{\hat{\rho}}\right) - \left(y_i^{*\hat{\rho}}\right)^{\frac{\hat{\rho}-1}{\hat{\rho}}}\left(y_i'^{\hat{\rho}}\right)^{\frac{1}{\hat{\rho}}}\right] + P_{ik}(m^*)\left(y_i^* - y_i'\right) \geq \left(x_i' - x_i^*\right),$$

which, by assumption 5, is true if

$$\alpha_i' \left[\left(\frac{\hat\rho - 1}{\hat\rho} y_i^{*\hat\rho} + \frac{1}{\hat\rho} y_i'^{\hat\rho} \right) - \left(y_i^{*\hat\rho} \right)^{\frac{\hat\rho-1}{\hat\rho}} \left(y_i'^{\hat\rho} \right)^{\frac{1}{\hat\rho}} \right] + P_{ik}(m^*) \left(y_i^* - y_i' \right) \geq \\ \gamma_i(m^*) \hat\rho \max \left\{ |y_i^* - y_i'|^{\hat\rho}, |y_i^* - y_i'|^{\frac{1}{\hat\rho}} \right\} \tag{20}$$

(the extra $\hat\rho$ before the maximizing operator is needed for a later step). But the term in square brackets is the difference between the weighted arithmetic mean and the weighted geometric mean of the two points $y_i^{*\hat\rho}$ and $y_i'^{\hat\rho}$; by the AM-GM inequality this difference is positive. Thus, there is some finite α_i' at which inequality (20) is true. Similarly, there is some finite α_i'' at which the expression $u_i(x_i^*, y_i^*) - u_i(x_i'', y_i'') \geq 0$ is true. Let $\alpha_i(m^*) = \max\{\alpha_i', \alpha_i''\}$ and now fix $\phi_i(m^*) = (\alpha_i(m^*), \beta_i(m^*))$.

Suppose that $y_i' < y_i''$ (the proof for the case where $y_i'' < y_i'$ is symmetric) and pick any $y_i \geq y_i''$. Suppose that

$$\alpha_i(m^*) \left[\left(\frac{\hat\rho - 1}{\hat\rho} y_i^{*\hat\rho} + \frac{1}{\hat\rho} y_i^{\hat\rho} \right) - \left(y_i^{*\hat\rho} \right)^{\frac{\hat\rho-1}{\hat\rho}} \left(y_i^{\hat\rho} \right)^{\frac{1}{\hat\rho}} \right] + P_{ik}(m^*) \left(y_i^* - y_i \right) \\ - \gamma_i(m^*) \hat\rho \max \left\{ |y_i^* - y_i|^{\hat\rho}, |y_i^* - y_i|^{\frac{1}{\hat\rho}} \right\} \geq 0, \tag{21}$$

which is true for $y_i = y_i''$ (see inequality (20)). Then the derivative of the left-hand side of this inequality is positive, implying that the inequality is true *for all* $y_i \geq y_i''$; to see this, take the derivative of the left-hand side and multiply by $(y_i - y_i^*) > 0$ to get either

$$\alpha_i(m^*) \left[y_i^{*\hat\rho} - y_i^{*\hat\rho-1} y_i + y_i^{\hat\rho} - y_i^* y_i^{\hat\rho-1} \right] + P_{ik}(m^*)(y_i^* - y_i) - \gamma_i(m^*) \hat\rho (y_i - y_i^*)^{\hat\rho} \tag{22}$$

or

$$\alpha_i(m^*) \left[y_i^{*\hat\rho} - y_i^{*\hat\rho-1} y_i + y_i^{\hat\rho} - y_i^* y_i^{\hat\rho-1} \right] + P_{ik}(m^*)(y_i^* - y_i) - \gamma_i(m^*) \frac{1}{\hat\rho} (y_i - y_i^*)^{1/\hat\rho}. \tag{23}$$

In either case, the expression is greater than the left-hand side of (21) because

$$\left[y_i^{*\hat\rho} - y_i^{*\hat\rho-1} y_i + y_i^{\hat\rho} - y_i^* y_i^{\hat\rho-1} \right] \geq \left[\left(\frac{\hat\rho - 1}{\hat\rho} y_i^{*\hat\rho} + \frac{1}{\hat\rho} y_i^{\hat\rho} \right) - \left(y_i^{*\hat\rho} \right)^{\frac{\hat\rho-1}{\hat\rho}} \left(y_i^{\hat\rho} \right)^{\frac{1}{\hat\rho}} \right]$$

reduces to

$$\left(\frac{\hat\rho - 1}{\hat\rho} y_i^{*\hat\rho} + \frac{1}{\hat\rho} y_i^{\hat\rho} \right) \geq \left(y_i^{*\hat\rho} \right)^{\frac{\hat\rho-1}{\hat\rho}} \left(y_i^{\hat\rho} \right)^{\frac{1}{\hat\rho}},$$

which is just the AM-GM inequality again. Thus, both (22) and (23) are positive. By continuity, (21) is positive for all $y_i \geq y_i''$ and so deviations resulting in $y_i \geq y_i''$ are not profitable. A symmetric argument shows that deviations to $y_i \leq y_i'$ are also not profitable. Since we already know that deviations resulting in $y_i \in (y_i', y_i'')$ are unprofitable, the proof is complete. □

Theorems 2 and 3

Theorem 2 follows from theorem 3 so we only prove the latter here.

Proof of theorem 3. For any $\theta \in \Theta$ let $p_i(\theta)$ be agent i's price for good y_i at the Walrasian or Lindahl equilibrium for environment θ. For any $m \in \nu(\Theta)$ let $\phi(m) \in \Theta$ identify an environment θ for which m is an equilibrium. Thus, $p_i(\phi(m))$ is the Walrasian or Lindahl price that must be charged to agent i in the environment $\phi(m)$. Pick any regular equilibrium point $m^* = (r^*, s^*)$ in $\mathcal{M}^*$ and, for notational simplicity, let $y_i^* = y_i(r^*)$ and $x_i^* = x_i(m^*)$. The proof then follows from three important observations that must be true at m^* for each $i \in \mathcal{I}$:

1. Because m^* is a Nash equilibrium for some $\theta \in \Theta$ the following first-order condition is satisfied for each $k \in \{1, \dots, J_i\}$:

$$\frac{\partial u_i(x_i^*, y_i^*|\theta_i)}{\partial y_i} \frac{\partial y_i(r_i^*)}{\partial r_{ik}} = \frac{\partial u_i(x_i^*, y_i^*|\theta_i)}{\partial x_i} \left[-\frac{\partial x_i(r^*, s^*)}{\partial r_{ik}} \right]. \tag{24}$$

2. If m^* maps to a Walrasian or Lindahl equilibrium for some $\theta \in \Theta$ then it must be that the transfers collected by the mechanism equals the transfers of the numéraire required by the Walrasian or Lindahl equilibrium:

$$x_i(r^*, s^*) = -p_i(\phi(r^*, s^*)) y_i(r^*). \tag{25}$$

3. If m^* maps to a Walrasian or Lindahl equilibrium for some $\theta \in \Theta$ then the Walrasian or Lindahl price must equal the marginal rate of substitution of y_i in terms of x_i:

$$\frac{\partial u_i(x_i^*, y_i^*|\theta_i)/\partial y_i}{\partial u_i(x_i^*, y_i^*|\theta_i)/\partial x_i} = p_i(\phi(r^*, s^*)). \tag{26}$$

Dividing both sides of (24) by $\partial u_i / \partial x_i$, inserting equation (26), and rearranging gives

$$\frac{\partial x_i(r^*, s^*)}{\partial r_{ik}} = -p_i(\phi(r^*, s^*)) \frac{\partial y_i(r^*)}{\partial r_{ik}}. \tag{27}$$

for each i and k.

Since (r^*, s^*) is a regular equilibrium point, differential changes in r_i accompanied by the requisite change in s_i lead to other points in $\mathcal{M}^*$ at which the above equations hold. Now take the total derivative of (25) with respect to r_i (allowing for the adjustment in s_i); since s_i maximizes x_i the envelope theorem guarantees that $dx_i/dr_{ik} = \partial x_i / \partial r_{ik}$ and so the total derivative is

$$\frac{\partial x_i(r^*, s^*)}{\partial r_{ik}} = -p_i(\phi(r^*, s^*)) \frac{\partial y_i(r^*)}{\partial r_{ik}} - \frac{dp_i(\phi(r^*, s^*))}{dr_{ik}} y_i(r^*). \tag{28}$$

Comparing equations (27) and (28), it must be that either $y_i(r^*) = 0$ or $dp_i(\phi(r^*, s^*))/dr_{ik} = 0$ for all k.

If $y_i(r^*) \neq 0$ but $dp_i(\phi(r^*, s^*))/dr_{ik} = 0$ for all k then, by (25),

$$g_i(r^*, s^*) = [p_i(\phi(r^*, s^*)) - q_i(r^*, s^*)]\, y_i(r^*)$$

and so $g_i(r^*, s^*)$ can be expressed as $h_i(r^*, s^*)y_i(r^*)$ for some function h_i such that $dh_i/dr_{ik} = 0$ for all k. But then $x_i(r^*, s^*)$ can be re-written as:

$$x_i(r^*, s^*) = -\left[q_i(r^*, s^*) + h_i(r^*, s^*)\right] y_i(r^*).$$

Label the bracketed term as $\tilde{q}_i(r^*, s^*)$ and we have that

$$x_i(r^*, s^*) = -\tilde{q}_i(r^*, s^*)y_i(r^*)$$

with $d\tilde{q}/dr_{ik} = 0$, giving the result.

If $y_i(r^*) = 0$ then by equation (25) we have $g_i(r^*, s^*) = 0$. It remains to show that $dq_i(r^*, s^*)/dr_{ik} = 0$. Since dy_i/dr_{ik} is bounded away from zero any perturbation of r_{ik} leads to $y_i \neq 0$; by regularity of (r^*, s^*), small perturbations lead to other regular equilibria with $y_i \neq 0$ at which $dq_i/dr_{ik} = 0$. Since q_i is continuously differentiable it must be that $dq_i(r^*, s^*)/dr_{ik} = 0$ as well. $\square$

Lemma 1

Lemma 1. A continuously-differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping (with constant ξ) if there exists $\xi \in (0, 1)$ such that $\sum_{k=1}^{n} \left| \frac{\partial f_i(x)}{\partial x_k} \right| \leq \xi < 1$ for all $x \in \mathbb{R}^n$ and all $i = 1, \ldots, n$. This property is also necessary for f to be a contraction mapping in the sup-norm.

Proof. First, we establish sufficiency. Take any $x, x' \in \mathbb{R}$, and let $u = x' - x$. Define $g(t) = f_i(x + ut)$ for $t \in \mathbb{R}$, and note

$$g'(t) = \sum_{k=1}^{n} \frac{\partial f_i(x + ut)}{\partial x_k} u_k.$$

Then,

$$\begin{aligned}
|f_i(x') - f_i(x)| &= |g(1) - g(0)| \\
&= \left| \int_0^1 \sum_{k=1}^{n} \frac{\partial f_i(x+ut)}{\partial x_k} u_k \, dt \right| \\
&\leq \int_0^1 \sum_{j=1}^{n} \left| \frac{\partial f_i(x+ut)}{\partial x_j} \right| \max_k |u_k| dt \\
&\leq \int_0^1 \xi \max_k |x'_k - x_k| dt \\
&= \xi d(x, x').
\end{aligned}$$

Since this inequality holds for all $i = 1, \ldots, n$, it implies $d(f(x'), f(x)) = \max_i |f_i(x') - f_i(x)| \leq \xi d(x, x')$. Second, we prove necessity. Suppose that for each $\xi \in (0,1)$, there exists $x^* \in \mathbb{R}^n$ such that $\sum_{k=1}^{n} \left| \frac{\partial f_i(x^*)}{\partial x_k} \right| > \xi$. Since f is continuously-differentiable, $\sum_{k=1}^{n} \left| \frac{\partial f_i(x')}{\partial x_k} \right| > \xi$ for all x' in a neighborhood $\mathcal{U}$ of x^*. For $\epsilon > 0$, define x^ϵ as $x_k^\epsilon = x_k^* + \text{sgn}\left\{ \frac{\partial f_i(x^*)}{\partial x_k} \right\} \epsilon$ for each $k = 1, \ldots, n$. Choose $\epsilon > 0$ such that $x^\epsilon \in \mathcal{U}$. Letting $u = x^\epsilon - x^*$, we have

$$\begin{aligned}
|f_i(x') - f_i(x^*)| &= \left| \int_0^1 \sum_{k=1}^{n} \frac{\partial f_i(x+ut)}{\partial x_k} u_k \, dt \right| \\
&= \int_0^1 \sum_{j=1}^{n} \left| \frac{\partial f_i(x+ut)}{\partial x_j} \right| \epsilon \, dt \\
&> \xi \epsilon = \xi d(x, x').
\end{aligned}$$

Therefore, for each $\xi \in (0,1)$, there exist x^* and x' such that $d(f(x'), f(x^*)) > \xi d(x', x^*)$, that is, the function is not a contraction mapping. □

Theorem 4

Proof of Theorem 4. We know that there cannot exist *any* one-dimensional mechanism that Nash implements the Walrasian correspondence, contractive or not. For the public goods setting, suppose by way of contradiction that the mechanism $(y, (q_i, g_i)_{i=1}^n)$ Nash implements the Lindahl correspondence and is contractive. By Theorem 3 we know that $g_i \equiv 0$ and so for any quasilinear environment

$$u_i(x_i, y|\theta_i) = v_i(y|\theta_i) + x_i$$

with $v_i'' \in [-M, 0)$ we have

$$U_i(r_i, r_{-i}) = v_i(y(r)|\theta_i) - q_i(r_{-i})y(r).$$

Agent i's best-response is given by $\rho_i(r_{-i})$ and satisfies the first-order condition

$$v_i'(y(\rho_i, r_{-i})|\theta_i) = q_i(r_{-i})$$

for all r_{-i}. Take any r^* and θ for which r^* is a Nash equilibrium at θ. By the Implicit Function Theorem the slope of ρ_i at r^* with respect to each r_j is shown to be

$$\frac{\partial \rho_i}{\partial r_j} = \frac{\partial q_i/\partial r_j - v_i''(y|\theta_i)\,\partial y/\partial r_j}{v_i''(y|\theta_i)\,\partial y/\partial r_i}.$$

For the mechanism to be contractive it is necessary (though not sufficient) that for all i and $j \neq i$

$$\left| \frac{\partial y/\partial r_j}{\partial y/\partial r_i} - \frac{\partial q_i/\partial r_j}{v_i''(y|\theta_i)\partial y/\partial r_i} \right| < 1. \tag{29}$$

Now select the agent j^* such that $|\partial y(r^*)/\partial r_{j^*}| \geq |\partial y(r^*)/\partial r_i|$ for all i. In order to satisfy equation 29 it must be that $\partial y/\partial r_{j^*}$ and $\partial q_i/\partial r_{j^*}$ have the opposite sign for all i and that $\partial q_i/\partial r_{j^*} \neq 0$ for all i. Therefore, each $\partial q_i/\partial r_{j^*}$ has the same sign for all $i \neq j^*$ (and $\partial q_{j^*}/\partial r_{j^*} = 0$) so that $\sum_i \partial q_i/\partial r_{j^*} \neq 0$. But since all r are Nash equilibria for some θ and the mechanism implements Lindahl allocations it must be that $\sum_i q_i(r_{-i}) = \kappa$ for all r and, therefore, that $\sum_i \partial q_i/\partial r_{j^*} = 0$; this is a contradiction.

□

Theorem 5

Proof of theorem 5. **Step 1: Proving the contraction property**

To show how to construct contractive mechanisms more generally, we begin the proof using arbitrary y, q_i, and g_i functions; we insert the specific functions given in equations when necessary.

In a two-dimensional mechanism agent i's utility function at θ defined over strategy profiles (r, s) is given by

$$U_i\left(r, s|\theta_i\right) = v_i\left(y\left(r\right)|\theta_i\right) - q_i\left(r_{-i}, s_{-i}\right) y\left(r\right) - g_i\left(r, s\right).$$

Henceforth we drop the dependence on θ for notational brevity. For each i and (r_{-i}, s_{-i}) define $\rho_i\left(r_{-i}, s_i\right) \in \mathcal{R}_i$ and $\sigma_i\left(r_{-i}, s_{-i}\right) \in \mathcal{S}_i$ to be i's best-response messages. Using the Implicit Function

Theorem, we solve for the slope of each ρ_i and σ_i by differentiating the identities

$$\frac{\partial U_i\left(\rho_i, \sigma_i, r_{-i}, s_{-i}\right)}{\partial r_i} \equiv \frac{\partial U_i\left(\rho_i, \sigma_i, r_{-i}, s_{-i}\right)}{\partial s_i} \equiv 0$$

with respect to each r_j and s_j. The resulting system of equations (for each i and j) is given by

$$\begin{bmatrix} \frac{\partial^2 U_i}{\partial r_i^2} & \frac{\partial^2 U_i}{\partial r_i \partial s_i} \\ \frac{\partial^2 U_i}{\partial s_i \partial r_i} & \frac{\partial^2 U_i}{\partial s_i^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \rho_i}{\partial r_j} & \frac{\partial \rho_i}{\partial s_j} \\ \frac{\partial \sigma_i}{\partial r_j} & \frac{\partial \sigma_i}{\partial s_j} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 U_i}{\partial r_i r_j} & \frac{\partial^2 U_i}{\partial r_i s_j} \\ \frac{\partial^2 U_i}{\partial s_i r_j} & \frac{\partial^2 U_i}{\partial s_i s_j} \end{bmatrix},$$

which has a unique solution if the left-most matrix is invertible. In that case the inverse is given by

$$\left(\frac{\partial^2 U_i}{\partial r_i^2}\frac{\partial^2 U_i}{\partial s_i^2} - \left(\frac{\partial^2 U_i}{\partial r_i \partial s_i}\right)^2\right)^{-1} \begin{bmatrix} \frac{\partial^2 U_i}{\partial s_i^2} & -\frac{\partial^2 U_i}{\partial r_i \partial s_i} \\ -\frac{\partial^2 U_i}{\partial r_i \partial s_i} & \frac{\partial^2 U_i}{\partial r_i^2} \end{bmatrix},$$

and so invertibility requires that for each i and j,

$$\left(\frac{\partial^2 U_i}{\partial r_i \partial s_i}\right)^2 \neq \frac{\partial^2 U_i}{\partial r_i^2}\frac{\partial^2 U_i}{\partial s_i^2}.$$

Letting $y(r)$ be linear (so that $y(r) = \sum_i \alpha_i r_i$ for some $(\alpha_1, \ldots, \alpha_n) > 0$), the relevant second derivatives for solving this system are

$$\begin{aligned}
\frac{\partial^2 U_i}{\partial r_i \partial r_j} &= v_i''(y(r))\alpha_i \alpha_j - \alpha_i \frac{\partial q_i}{\partial r_j} - \frac{\partial^2 g_i}{\partial r_i \partial r_j}, \\
\frac{\partial^2 U_i}{\partial r_i \partial s_i} &= -\frac{\partial^2 g_i}{\partial r_i \partial s_i}, \\
\frac{\partial^2 U_i}{\partial r_i \partial s_j} &= -\alpha_i \frac{\partial q_i}{\partial s_j} - \frac{\partial^2 g_i}{\partial r_i \partial s_j}, \\
\frac{\partial^2 U_i}{\partial r_i^2} &= v_i''(y(r))\alpha_i^2 - \frac{\partial^2 g_i}{\partial r_i^2}, \\
\frac{\partial^2 U_i}{\partial s_i \partial r_j} &= -\frac{\partial^2 g_i}{\partial s_i \partial r_j}, \\
\frac{\partial^2 U_i}{\partial s_i^2} &= -\frac{\partial^2 g_i}{\partial s_i^2},
\end{aligned}$$

and

$$\frac{\partial^2 U_i}{\partial s_i \partial s_j} = -\frac{\partial^2 g_i}{\partial s_i \partial s_j}.$$

Note that if $\partial^2 g_i / \partial r_i^2 \geq 0$ and $\partial^2 g_i / \partial s_i^2 > 0$ then the utility function is strictly concave, so the best-response is always unique.

With these partial derivatives, the invertibility conditions become

$$-v_i''\left(y\left(r\right)\right) \neq \frac{\left(\frac{\partial^2 g_i}{\partial r_i \partial s_i}\right)^2 - \frac{\partial^2 g_i}{\partial r_i^2}\frac{\partial^2 g_i}{\partial s_i^2}}{\alpha_i^2 \frac{\partial^2 g_i}{\partial s_i^2}}$$

and the slopes of the best-response functions are given by

$$\begin{bmatrix} \frac{\partial \rho_i}{\partial r_j} & \frac{\partial \rho_i}{\partial s_j} \\ \frac{\partial \sigma_i}{\partial r_j} & \frac{\partial \sigma_i}{\partial s_j} \end{bmatrix} = \left(\frac{\partial^2 U_i}{\partial r_i^2}\frac{\partial^2 U_i}{\partial s_i^2} - \left(\frac{\partial^2 U_i}{\partial r_i \partial s_i}\right)^2\right)^{-1} \begin{bmatrix} \frac{\partial^2 U_i}{\partial r_i \partial s_i}\frac{\partial^2 U_i}{\partial s_i r_j} - \frac{\partial^2 U_i}{\partial s_i^2}\frac{\partial^2 U_i}{\partial r_i r_j} & \frac{\partial^2 U_i}{\partial r_i \partial s_i}\frac{\partial^2 U_i}{\partial s_i s_j} - \frac{\partial^2 U_i}{\partial s_i^2}\frac{\partial^2 U_i}{\partial r_i s_j} \\ \frac{\partial^2 U_i}{\partial r_i \partial s_i}\frac{\partial^2 U_i}{\partial r_i r_j} - \frac{\partial^2 U_i}{\partial r_i^2}\frac{\partial^2 U_i}{\partial s_i r_j} & \frac{\partial^2 U_i}{\partial r_i \partial s_i}\frac{\partial^2 U_i}{\partial r_i s_j} - \frac{\partial^2 U_i}{\partial r_i^2}\frac{\partial^2 U_i}{\partial s_i s_j} \end{bmatrix}.$$

We now insert our particular y, q_i, and g_i functions. Specifically, let

$$y\left(r\right) = \sum_i r_i,$$

$$q_i\left(r_{-i}, s_{-i}\right) = \frac{\kappa}{n} + \frac{1}{\delta}\left(r_{i-1} - r_{i+1}\right) + \delta\frac{n-1}{n^2}\left(s_{i-1} - \frac{1}{n}r_{i+1}\right),$$

and

$$g_i\left(r, s\right) = \frac{1}{2}\left(s_i - \frac{1}{n}r_{i+1}\right)^2 + \frac{\delta}{2}\left(s_{i-1} - \frac{1}{n}r_i\right)^2,$$

which gives a best response function of $\sigma_i\left(r_{-i}, s_{-i}\right) = r_{i+1}/n$. Clearly, at any best response point (and, hence, any equilibrium point), $g_i\left(r, s\right) = 0$. Also, we can calculate the slopes of σ_i directly: $\partial\sigma_i/\partial r_{i+1} \equiv 1/n$, $\partial\sigma_i/\partial r_j \equiv 0$ for $j \neq i+1$, and $\partial\sigma_i/\partial s_j \equiv 0$ for all j.

The non-zero second partial derivatives of g_i are

$$\frac{\partial^2 g_i}{\partial r_i^2} = \delta\frac{1}{n^2},$$
$$\frac{\partial^2 g_i}{\partial r_i \partial s_{i-1}} = -\delta\frac{1}{n},$$
$$\frac{\partial^2 g_i}{\partial s_i^2} = 1,$$

and

$$\frac{\partial^2 g_i}{\partial s_i \partial r_{i+1}} = -\frac{1}{n}$$

(all other second-partial derivatives of g_i are identically zero). The non-zero derivatives of q_i are

$$\frac{\partial q_i}{\partial r_{i-1}} = \frac{1}{\delta},$$
$$\frac{\partial q_i}{\partial r_{i+1}} = -\frac{1}{\delta} - \delta\frac{n-1}{n^3},$$

and

$$\frac{\partial q_i}{\partial s_{i-1}} = \delta \frac{n-1}{n^2}.$$

To calculate the slopes of ρ_i we need to appeal to the general Implicit Function Theorem argument above. The invertibility condition reduces to

$$v_i''\left(y\left(r\right)\right) \neq \frac{\delta}{n^2},$$

which holds for positive δ since $v_i'' < 0$.

Plugging the derivatives of q_i and g_i into the formulas above gives

$$\begin{aligned}
\frac{\partial \rho_i}{\partial r_{i-1}} &= \frac{v_i''(y(r)) - \frac{1}{\delta}}{-v_i''(y(r)) + \delta \frac{1}{n^2}}, \\
\frac{\partial \rho_i}{\partial r_{i+1}} &= \frac{v_i''(y(r)) + \frac{1}{\delta} + \delta \frac{n-1}{n^3}}{-v_i''(y(r)) + \delta \frac{1}{n^2}},
\end{aligned}$$

and

$$\frac{\partial \rho_i}{\partial r_j} = \frac{v_i''(y(r))}{-v_i''(y(r)) + \delta \frac{1}{n^2}}$$

for all $j \notin \{i-1, i+1\}$; and

$$\frac{\partial \rho_i}{\partial s_{i-1}} = \frac{\delta \frac{1}{n^2}}{-v_i''(y(r)) + \delta \frac{1}{n^2}}$$

and $\partial \rho_i / \partial s_j = 0$ for all $j \neq i-1$. As a check, one can use the Implicit Function Theorem method to verify that $\partial \sigma_i / \partial r_{i+1} = 1/n$, $\partial \sigma_i / \partial s_{i+1} = 0$, and $\partial \sigma_i / \partial r_j = \partial \sigma_i / \partial s_j = 0$ for all $j \neq i+1$, as was derived directly above.

As δ grows large we have that $\partial \rho_i / \partial r_{i+1}$ converges to $(n-1)/n$, $\partial \rho_i / \partial r_j$ converges to zero for all $j \neq i+1$, $\partial \rho_i / \partial s_{i-1}$ converges to one, and $\partial \rho_i / \partial s_j$ converges to zero for all $j \neq i-1$. Therefore,

$$\begin{aligned}
\lim_{\delta \to \infty} \left(\sum_{j \neq i} \left| \frac{\partial \rho_j}{\partial r_i} \right| + \sum_{j \neq i} \left| \frac{\partial \sigma_j}{\partial r_i} \right| \right) &= \lim_{\delta \to \infty} \left(\left| \frac{\partial \rho_{i-1}}{\partial r_i} \right| + \left| \frac{\partial \sigma_{i-1}}{\partial r_i} \right| \right) \\
&= \left| \frac{n-1}{n} \right| + \left| \frac{1}{n} \right| \\
&= 1
\end{aligned}$$

and

$$\lim_{\delta\to\infty}\left(\sum_{j\neq i}\left|\frac{\partial\rho_j}{\partial s_i}\right|+\sum_{j\neq i}\left|\frac{\partial\sigma_j}{\partial s_i}\right|\right)=\lim_{\delta\to\infty}\left|\frac{\partial\rho_{i+1}}{\partial s_i}\right|$$
$$=1.$$

If we can verify that both of these sums approach one from below then for large but finite values of δ the mechanism will be contractive.

For the first condition we know that $|\partial\sigma_{i-1}/\partial r_i| = 1/n$ for all δ, so it suffices to check that $|\partial\rho_{i-1}/\partial r_i|$ converges to $|(n-1)/n|$ from below. This is true if

$$\left|\frac{v''_{i-1}(y(r))+\frac{1}{\delta}+\delta\frac{n-1}{n^3}}{-v''_{i-1}(y(r))+\delta\frac{1}{n^2}}\right|<\left|\frac{n-1}{n}\right|$$

for large but finite δ. By rearranging this reduces to

$$\left|v''_{i-1}(y(r))+\frac{1}{\delta}+\delta\frac{n-1}{n^3}\right|<-\frac{n-1}{n}v''_{i-1}(y(r))+\delta\frac{n-1}{n^3}.$$

If δ is large (specifically, if

$$\delta>\frac{\left(B+\sqrt{B^2-4\frac{n-1}{n^3}}\right)n^3}{2(n-1)} \tag{30}$$

or if $B^2<4(n-1)/n^3$) then the term inside the absolute value becomes positive, so that we have

$$v''_{i-1}(y(r))+\frac{1}{\delta}+\delta\frac{n-1}{n^3}<-\frac{n-1}{n}v''_{i-1}(y(r))+\delta\frac{n-1}{n^3},$$

or

$$\frac{1}{\delta}<-\frac{2n-1}{n}v''_{i-1}(y(r)).$$

Since v''_{i-1} is bounded away from zero this inequality is satisfied for large enough δ (specifically, for $\delta>n/(2B(n-1))$, but this is implied by the previous lower bound on δ whenever $B\geq 4(n-1)/n^3$).

For the second condition we check that $\partial\rho_{i+1}/\partial s_i$ converges to one from below but does not equal one for finite δ; but this is true for all $\delta>0$ since

$$\frac{\partial\rho_{i+1}}{\partial s_i}=\frac{\delta\frac{1}{n^2}}{-v''_{i+1}(y(r))+\delta\frac{1}{n^2}}$$

and v''_{i+1} is negative and bounded away from zero.

Thus, for δ large enough to satisfy inequality (30) and $\delta>n/(2B(n-1))$ the best response mapping is a contraction mapping.

Step 2: Proving that the mechanism implements Lindahl allocations.

To see that a unique Lindahl equilibrium exists for all θ, note that the three necessary and sufficient conditions for any Lindahl equilibrium are:

1. Given p_i^* and y^*, it must be that $x_i^* = -p_i^* y^*$ for all i;
2. This implies that that $\partial v_i(y^*)/\partial y = p_i^*$ for each i; and
3. Linearity of the firm's profit function then implies that $\sum_i \partial v_i(y^*)/\partial y = \sum_i p_i^* = \kappa$.

Using these conditions we can derive the unique equilibrium in three steps:

1. Since $v_i'' \in (-B, -1/B)$ for each i there is one unique y^* satisfying the third necessary condition;
2. Given the unique y^*, there is one unique p_i^* for each i satisfying the second condition; and
3. Given y^* and p_i^* there is one unique x_i^* for each i satisfying the first condition.

Since the mechanism is contractive it also has a unique Nash equilibrium (r^*, s^*) for every θ. Now take the equilibrium message (r^*, s^*) and let $p_i^* = q_i(r_{-i}^*, s_{-i}^*)$. Then $x_i(r^*, s^*) = -p_i^* y(r^*)$ for each i, satisfying the first condition. Furthermore, the first-order condition for maximization in r_i at an equilibrium point imply that

$$v_i'(y(r^*))\ \frac{\partial y_i(r^*)}{\partial r_i} = p_i^* \frac{\partial y_i(r^*)}{\partial r_i} + \frac{\partial g_i(r^*, \sigma(r^*))}{\partial r_i},$$

but since $\partial y_i(r)/\partial r_i \neq 0$ and $\partial g_i/\partial r_i = 0$ at the equilibrium point implies that $v_i'(y(r^*)) = p_i^*$, satisfying the second condition. Finally, it is easy to check that $\sum_i q_i(r_{-i}^*, s_{-i}^*) = \kappa$ at the equilibrium point since $s_i^* = r_{i+1}^*/n$ and so the third condition is satisfied. Thus, the unique equilibrium point is equal to the unique Lindahl allocation. □

Theorem ??

Proof. It is enough to show that the Walrasian correspondence cannot be implemented by a contractive mechanism in the case where there is one private good and one numeraire good. We establish the result for a two-dimensional mechanism, arguing then that additional dimensions would not help. By way of contradiction, consider any mechanism Γ such that

1. Each agent i's message space is $\mathcal{M}_i = \mathcal{R}_i \times \mathcal{S}_i$ where $\mathcal{R}_i = \mathcal{S}_i = \mathbb{R}$, and
2. The outcome function is $(y_i(r, s_{-i}), x_i(r, s))_{i \in \mathcal{I}}$ where:

$$x_i(s, r) = -q_i(r_{-i}, s_{-i}) y_i(r, s_{-i}) - g_i(r, s),$$

for some functions q_i and g_i,

and assume that this mechanism is contractive and implements the Walrasian correspondence. Agent i's utility function in the mechanism is given by:

$$u_i(m) = v_i(y_i(r, s_{-i})) - q_i(r_{-i}, s_{-i})y_i(r, s_{-i}) - g_i(r, s).$$

Recall that

$$\sigma_i(r, s_{-i}) := \arg\max_{s_i' \in \mathcal{S}_i} \; -g_i(r, s)$$

and

$$\sigma(r) := \{s^* \in \mathcal{S} : (\forall i \in \mathcal{I}) s_i^* \in \sigma_i(r, s_{-i}^*)\}.$$

Step 1. We demonstrate that $\sigma_i(r)$ satisfies $\sum_{j=1}^n \left|\frac{\partial \sigma_i(r)}{\partial r_j}\right| \leq 1$ for all r that satisfy the first-order conditions. Since $\sigma_i(r_{-i}, s_{-i}) = \sigma_i(\rho_i(r_{-i}, s_{-i}), r_{-i}, s_{-i})$ is a contraction mapping, it follows from Lemma 1 that

$$\sum_{j\neq i}\left|\frac{\partial \sigma_i}{\partial r_j}\right| + \sum_{j\neq i}\left|\frac{\partial \sigma_i}{\partial s_j}\right| + \left|\frac{\partial \sigma_i}{\partial r_i}\right|\left(\sum_{j\neq i}\left|\frac{\partial \rho_i}{\partial r_j}\right| + \sum_{j\neq i}\left|\frac{\partial \rho_i}{\partial s_j}\right|\right) < 1. \tag{31}$$

Without knowing θ, hence without knowing $\rho_i(\cdot)$ and its exact slope (besides $\sum_{j=1}^n \left|\frac{\partial \rho_i}{\partial r_j}\right| + \sum_{j=1}^n \left|\frac{\partial \rho_i}{\partial r_j}\right| < 1$), (31) requires

$$\frac{|d|\sigma_i(r, s_{-i})}{|d|(r, s_{-i})} := \sum_{j=1}^n \left|\frac{\partial \sigma_i(r, s_{-i})}{\partial r_j}\right| + \sum_{j\neq i}\left|\frac{\partial \sigma_i(r, s_{-i})}{\partial s_j}\right| \leq 1, \tag{32}$$

for all $(r, s) \in \mathcal{M}^*$. Now we proceed by induction to prove the claim. Pick any i and k in $\mathcal{I}$, and define

$$\sigma_i(r, s_{-ik}) := \sigma_i(r, \sigma_k(r, \sigma_i(r, s_{-ik}), s_{-ik}), s_{-ik})$$

for all $(r, s) \in \mathcal{M}^*$. Note

$$\frac{|d|\sigma_i(r, s_{-ik})}{|d|(r, s_{-ik})} = \sum_{j=1}^n \left|\frac{\partial \sigma_i}{\partial r_j}\right| + \left|\frac{\partial \sigma_i}{\partial s_k}\right|\left(\sum_{j=1}^n \left|\frac{\partial \sigma_k}{\partial r_j}\right| + \left|\frac{\partial \sigma_k}{\partial s_i}\right| \frac{|d|\sigma_i(r, s_{-ik})}{|d|(r, s_{-ik})} + \sum_{\ell\neq i,k}\left|\frac{\partial \sigma_k}{\partial s_\ell}\right|\right) + \sum_{\ell\neq i,k}\left|\frac{\partial \sigma_i}{\partial s_\ell}\right|,$$

which is an equation of the form $a = \alpha a + \beta$ where $\alpha + \beta \leq 1$. Consequently, $\frac{|d|\sigma_i(r, s_{-ik})}{|d|(r, s_{-ik})} \leq 1$. Now take $j \neq i, k$, and define

$$\sigma_i(r, s_{-ijk}) := \sigma_i(r, \sigma_j(r, \sigma_i(r, s_{-ijk}), s_{-ijk}), s_{-ijk}).$$

A similar argument shows that $\sigma_i(r, s_{-ijk})$ is a contraction mapping in r. Proceeding inductively leads to the conclusion that $\sigma_{-i}(r)$ is contractive.

Step 2. Theorem 3 implies that for all $(r,s) \in \mathcal{M}^*$, $\frac{\partial g_i(r,s)}{\partial r_i} = 0$. As a result, the first-order conditions (w.r.t. each r_i) at $s_{-i} = \sigma_{-i}(r)$ are given by:

$$v_i'(y_i(r, \sigma_{-i}(r)))\frac{\partial y_i}{\partial r_i} = q_i(r_{-i}, \sigma_{-i}(r))\frac{\partial y_i}{\partial r_i}.$$

By strict convexity of preferences (so $v_i^{-1}(\cdot)$ exists),

$$y_i(\rho_i(r_{-i}), r_{-i}, \sigma_{-i}(\rho_i(r_{-i}), r_{-i})) = v_i'^{-1}(q_i(\rho_i(r_{-i}), r_{-i}, \sigma_{-i}(\rho_i(r_{-i}), r_{-i}))). \tag{33}$$

By definition, $\rho_i(r_{-i}) = \rho_i(r_{-i}, \sigma_{-i}(\rho_i(r_{-i}), r_{-i}))$. Since $\rho_i(r_{-i}, s_{-i})$ is assumed to be a contraction mapping, so is $\rho_i(r_{-i})$ by step 1 (indeed, it is essentially a composition of contraction mappings). From (33), and taking the derivative w.r.t. r_j, we obtain

$$\frac{\partial y_i}{\partial r_i}\frac{\partial \rho_i}{\partial r_j} + \frac{\partial y_i}{\partial r_j} + \sum_{k \neq i}\frac{\partial y_i}{\partial s_k}\left(\frac{\partial \sigma_k}{\partial r_i}\frac{\partial \rho_i}{\partial r_j} + \frac{\partial \sigma_k}{\partial r_j}\right) = \frac{dq_i/dr_j}{v_i''(\cdot)}$$

So,

$$\sum_{j \neq i}\left|\frac{\partial \rho_i(r_{-i})}{\partial r_j}\right| = \sum_{j \neq i}\left|\frac{dq_i/dr_j}{v_i''(v^{-1}(\cdot))} - \frac{\partial y_i}{\partial r_j} - \sum_{k \neq i}\frac{\partial y_i}{\partial s_k}\frac{\partial \sigma_k}{\partial r_j}\right|\left|\frac{1}{\frac{\partial y_i}{\partial r_i} + \sum_{k \neq i}\frac{\partial y_i}{\partial s_k}\frac{\partial \sigma_k}{\partial r_i}}\right| < 1,$$

which implies

$$\sum_{i=1}^{n}\left|\frac{\partial y_i}{\partial r_i} + \sum_{k \neq i}\frac{\partial y_i}{\partial s_k}\frac{\partial \sigma_k}{\partial r_i}\right| > \sum_{i=1}^{n}\sum_{j \neq i}\left|\frac{dq_i/dr_j}{v_i''(v^{-1}(\cdot))} - \frac{\partial y_i}{\partial r_j} - \sum_{k \neq i}\frac{\partial y_i}{\partial s_k}\frac{\partial \sigma_k}{\partial r_j}\right|. \tag{34}$$

Note that we can replace the price function q_i with δq_i, for any $\delta > 0$, and the mechanism remains walrasian. So, the contraction property (34) is equivalent to

$$\begin{aligned}
\sum_{i=1}^{n}\left|\frac{\partial y_i}{\partial r_i} + \sum_{k \neq i}\frac{\partial y_i}{\partial s_k}\frac{\partial \sigma_k}{\partial r_i}\right| \quad &> \quad \sum_{i=1}^{n}\sum_{j \neq i}\left|\frac{\partial y_i}{\partial r_j} + \sum_{k \neq i}\frac{\partial y_i}{\partial s_k}\frac{\partial \sigma_k}{\partial r_j}\right|. \\
&= \quad \sum_{j=1}^{n}\sum_{i \neq j}\left|\frac{\partial y_i}{\partial r_j} + \sum_{k \neq i}\frac{\partial y_i}{\partial s_k}\frac{\partial \sigma_k}{\partial r_j}\right|.
\end{aligned} \tag{35}$$

In words, (35) says that contraction requires functions y_i to be overall more sensitive to r_i than r_j. To satisfy the balance requirement of the Walrasian correspondence,

$$\sum_{i=1}^{n} y_i(r,s) = 0,$$

for all $(r,s) \in \mathcal{M}^*$, because $\nu(\Theta) = \mathcal{M}^*$ by Theorem 3. This implies

$$\frac{\partial}{\partial r_j} \sum_{i=1}^{n} y_i(r, \sigma_{-i}(r)) = 0,$$

for all r and $j = 1, \dots, n$, that is,

$$\sum_{i=1}^{n} \left(\frac{\partial y_i}{\partial r_j} + \sum_{k \neq i} \frac{\partial y_i}{\partial s_k} \frac{\partial \sigma_k}{\partial r_j} \right) = 0.$$

Equivalently,

$$\frac{\partial y_j}{\partial r_j} + \sum_{k \neq j} \frac{\partial y_j}{\partial s_k} \frac{\partial \sigma_k}{\partial r_j} = -\sum_{i \neq j} \left(\frac{\partial y_i}{\partial r_j} + \sum_{k \neq i} \frac{\partial y_i}{\partial s_k} \frac{\partial \sigma_k}{\partial r_j} \right).$$

But then,

$$\begin{aligned} \sum_{j=1}^{n} \left| \frac{\partial y_j}{\partial r_j} + \sum_{k \neq j} \frac{\partial y_j}{\partial s_k} \frac{\partial \sigma_k}{\partial r_j} \right| &= \sum_{j=1}^{n} \left| \sum_{i \neq j} \left(\frac{\partial y_i}{\partial r_j} + \sum_{k \neq i} \frac{\partial y_i}{\partial s_k} \frac{\partial \sigma_k}{\partial r_j} \right) \right|. \\ &\leq \sum_{j=1}^{n} \sum_{i \neq j} \left| \frac{\partial y_i}{\partial r_j} + \sum_{k \neq i} \frac{\partial y_i}{\partial s_k} \frac{\partial \sigma_k}{\partial r_j} \right|, \end{aligned} \tag{36}$$

which — when evaluated at $(\rho_i(r_{-i}), r_{-i})$— is in contradiction with (35). □

Theorem 6

Proof of theorem 6. [TO BE WRITTEN] □

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