

# Updating Toward the Signal<sup>\*</sup>

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## Abstract

Modelers frequently assume (either implicitly or explicitly) that an agent's posterior expectation of some variable lies between their prior mean and the realization of an unbiased signal of that variable. We call this property *updating toward the signal*, or *UTS*. We show that if the prior and signal error densities are both symmetric and quasiconcave then UTS will occur. If, for a given prior, UTS occurs for *all* symmetric and quasiconcave error densities then in fact the prior must be symmetric and quasiconcave. Similar characterizations are derived for two additional updating requirements that are strictly weaker than UTS.

*JEL*: C11, D01, D81, D83, D84

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## 1 Introduction

It is common in economic theory to assume that agents infer the value of some meaningful random variable based on imperfect observation. For example, bidders in common-value auctions infer the unknown value of an object from private signals (Wilson, 1977; Milgrom and Weber, 1982), oligopolists choose production levels after observing noisy signals of the demand curve or of their

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marginal costs (Novshek and Sonnenschein, 1982; Shapiro, 1986), producers in the Lucas “island” economy react to prices that serve as noisy signals of aggregate demand (Lucas, 1972), and agents search for the best price or wage when the underlying distribution is unknown (DeGroot, 1968; Rothschild, 1974).

The standard intuition in such settings is that, by Bayes’s rule, posterior beliefs should represent a weighted average of the prior beliefs and the observed signal. Thus, it is typically assumed that the posterior expectation is a convex combination of the prior expectation and the observed signal. We refer to this property as *updating toward the signal*, or *UTS*. The intuition of UTS is accurate for standard parametric examples such as a normal prior with a normally distributed and independent signal error. In general, however, UTS does not hold; Observation 5 below shows how any arbitrary combination of a prior mean, signal realization and posterior mean can occur when no restrictions are placed on the distributions.

Although many authors assume UTS implicitly by using normal distributions (Morris and Shin, 2002, 2006, e.g.), some make this assumption explicit (Shapiro, 1986; Moore and Healy, 2008) as it is necessary for the results they describe.<sup>3</sup> Naturally, assuming UTS means assuming some restrictions on the admissible distributions of the variable and the signal error. To the best of our knowledge, however, the strength and nature of these restrictions are not well understood. It is also not known how ‘far’ one can relax the assumption of normality while still guaranteeing UTS.

In this paper we identify conditions on the prior and error distributions sufficient to guarantee UTS. We also consider sufficient conditions for the weaker requirements of *updating in the direction of the signal (UDS)*, where the posterior mean must lie in the same direction as the signal (relative to the prior mean), and *mean reinforcement (MR)*, which requires that the posterior mean equal the prior mean when the signal equals the prior mean. We then provide necessary conditions on the prior distribution when UTS, UDS, or MR are required to hold for all error distributions in a certain family of admissible errors. These latter results can be interpreted as requirements on the prior that guarantee a certain degree of robustness of the posterior mean.

Throughout this paper we allow the error distribution to depend on the realizations of the underlying variable; however, we do require that this dependence be symmetric in the underlying variable. In other words, we require that the signal error distributions be identical for any two realizations of the underlying variable that are equidistant from the prior mean. We refer to this condition

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<sup>3</sup> Using normal distributions also guarantees other convenient properties that are not implied by UTS such as monotonicity of the posterior mean in the signal realization.

as *symmetric dependence*.

Under symmetric dependence, we find that MR obtains when the prior distribution and the (mean-zero) error distribution are both symmetric. If the error term is also quasiconcave then UDS is guaranteed. If, in addition, the errors are independent and the prior is quasiconcave then UTS is satisfied. We argue that these results are tight by showing examples where the conditions for one updating requirement (e.g., UDS) are satisfied but the stronger updating requirement (e.g., UTS) fails.

In the other direction, one would like to know the content of explicitly assuming UTS (or UDS or MR) in a theoretical model. To examine this we take the approach of a modeler who is free to choose a prior distribution but cannot commit to strong assumptions about the error distribution. This would be appropriate, for example, in a model of asset returns with market microstructure noise because the exact cause and structure of the noise in high-frequency asset return data is not well understood.<sup>4</sup>

Assuming that the family of admissible error distributions is a broad enough class of symmetric distributions (specifically, assuming the family contains all symmetric distributions that put a mass of one half on two different mass points), we show that if MR holds for all errors in this family then the prior must be symmetric. If the stronger condition of UDS is assumed then an impossibility result obtains: there is no prior distribution for which UDS can be guaranteed for all errors in the family. If the family of errors is a broad enough class of symmetric and quasiconcave distributions (specifically, assuming the family contains all symmetric uniform distributions), we show that MR again implies a symmetric prior, but now UDS places no additional restrictions; the two are equally strong assumptions. Finally, assuming UTS in this case implies that the prior must be both symmetric *and* quasiconcave.

The intuition for the necessity of symmetry (for MR and UDS) and of quasiconcavity (for UTS) is shown in Figure I. In panel A we show an asymmetric prior density  $f_X$  with mean  $\mu$ . If the error term has a uniform distribution with radius  $a > 0$ , then the posterior distribution after observing a signal  $z = \mu$  is given by the shaded area under the prior density function.<sup>5</sup> The

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<sup>4</sup> Zhou (1996), Andersen, Bollerslev, Diebold, and Labys (2000), and others show that, in practice, this microstructure noise creates a substantial bias when estimating the volatility of an asset using the realized variance of returns sampled at a high frequency. In response, Zhang, Mykland, and Ait-Sahalia (2005), Hansen and Lunde (2006), and Ait-Sahalia, Mykland, and Zhang (2005) explicitly model the microstructure noise as a random error term added to the underlying asset price—acknowledging that the structure of this noise is not well understood—and then derive the optimal estimation strategy under these conditions.

<sup>5</sup> Technically, the posterior distribution would be normalized to have a unit integral.

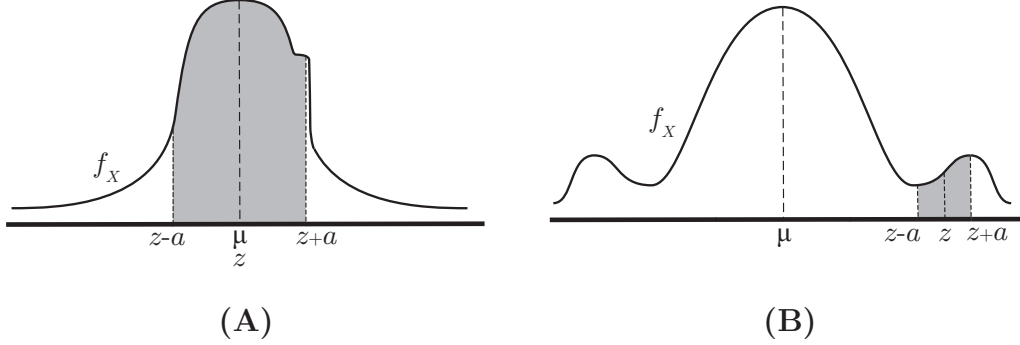


Fig. I. Demonstrating the necessary conditions for (A) MR and UDS, and (B) UTS.

asymmetry in the prior leads to an asymmetry in the posterior. This causes the posterior mean to be slightly greater than the prior mean, violating MR. If  $z$  were slightly less than  $\mu$  the posterior mean would still be greater than  $\mu$ , violating UDS. Following this argument for *every*  $a > 0$  leads to the conclusion that MR and UDS require a symmetric prior density function when all uniformly-distributed errors are admissible.

In panel (B) we show a prior distribution that is not quasiconcave. Again, the error distribution is uniform with radius  $a > 0$  and the posterior distribution after observing the signal  $z$  is given by the shaded area under the curve. Since the posterior distribution is strictly increasing over its domain, its mean must be greater than  $z$ , violating UTS. In general, if the signal realization  $z$  occurs in a region where quasiconcavity of the prior fails and if the radius of the uniform error distribution is sufficiently small then UTS will be violated. Thus, quasiconcavity is necessary for UTS when all uniformly-distributed errors are admissible.

The sufficiency results and the necessity results link up in a way that allows us to identify three characterization theorems. If the family of error terms is a sufficiently broad class of symmetric errors then symmetry of the prior is *equivalent* to requiring MR for the entire family of errors. If the family of error terms is a sufficiently broad class of symmetric and quasiconcave error terms then symmetry of the prior is equivalent to MR and, somewhat surprisingly, MR is equivalent to UDS. Finally, for the same family of symmetric and quasiconcave error terms, symmetry and quasiconcavity of the prior is equivalent to UTS.

In the next section we take two of our characterization theorems and apply them to a simple portfolio allocation problem. Since the location of an investor's beliefs about the expected return of a risky asset determine whether or not he will choose to invest in that asset, we can use UTS, UDS, and MR to ask which types of signals the investor might receive that would reinforce his prior decision to invest or not invest in the risky asset. Translated to this environment, our results say that if a broad family of symmetric and qua-

siconcave error terms is considered admissible then positive signals reinforce investors whose prior decision was to invest in the risky asset (and negative signals reinforce decision not to invest) if and only if the prior distribution on the asset's return is symmetric and quasiconcave. Thus, strong assumptions on the prior are needed to generate this sort of monotonicity in the interpretation of signals, and, in the other direction, assuming this monotonicity in the interpretation of the signal implies strong restrictions on the shape of the prior.

One view of the necessity results is that they provide a justification for using a normally-distributed prior; since symmetric and quasiconcave densities (such as the normal distribution) can guarantee UTS for a broad class of possible error terms, the assumption of UTS is fairly robust with normal priors. If the prior is not symmetric and quasiconcave (not 'normal-like') then there exists some uniformly-distributed error term and some signal realization such that UTS fails. In the portfolio choice problem, if the prior belief on asset returns is not 'normal-like' then exists a situation where an investor has a uniformly-distributed error term, would have invested before seeing the signal, and chose *not* to invest after observing a positive signal. If the modeler cannot rule out such error distributions then this 'non-monotonic' behavior cannot be ruled out.

Since the posterior mean can also be thought of as the Bayes estimator of the underlying variable using a quadratic loss function, the sufficient conditions for UTS can be also used by econometricians to guarantee reasonably well-behaved parameter estimates without relying on the standard normality assumptions. For example, the literature on robust Bayesian analysis (see Berger (1994) for an overview) argues that the family of symmetric and quasiconcave prior densities is particularly robust when performing Bayesian hypothesis tests; since we show that symmetric and quasiconcave priors imply and are implied by UTS (when the error distributions are also symmetric and quasiconcave), we provide another sense in which this family of priors is robust.<sup>6</sup>

Perhaps the closest paper to ours is the work of Andrews, Arnold, and Krutchkoff

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<sup>6</sup> There has also been some interest by statisticians in *credibility theory*—the use of posterior means that are (typically) assumed to be linear in the observed data to adjust parameter estimates in actuarial models—and its appropriateness (see Kahn, 1975 for an introduction). The capstone of this literature is Diaconis and Ylvisaker (1979), in which the authors characterize a conjugate family of distributions by requiring linearity of the posterior expectation. Our results are independent of this literature, however, since linearity of the posterior expectation is neither implied by nor implies UTS. Furthermore, the statistics literature has focused on implied restrictions on the parameters of certain families of distributions rather than restrictions on the density function directly, as provided by our results.

(1972). Using a normally-distributed error distribution and a symmetric prior they show that mean reinforcement (MR) is satisfied and that if the prior is also quasiconcave then UTS is satisfied.<sup>7</sup> Since normal densities are symmetric and quasiconcave, these two results are corollaries of our Propositions 6 and 9, respectively.

In the economics literature this paper is similar in spirit to Milgrom (1981). Milgrom requires that higher signal realizations lead to higher posterior beliefs in the sense of first-order stochastic dominance. He uses a similar robustness approach—requiring monotonicity for all non-degenerate priors—and shows that a signal distribution has the strict monotone likelihood ratio property (MLRP) if and only if monotonicity holds for all priors. Our paper differs from Milgrom’s in that we seek to order the posterior mean relative to the prior mean while Milgrom’s goal is to order various posterior distributions according to their signal realizations; in fact, the UTS concept is neither implied by nor implies any sort of monotonicity of the posterior mean. Additionally, Milgrom’s approach fixes an error distribution and considers the family of all non-degenerate priors, while we fix a prior and consider various families of error distributions. A systematic study of stochastic dominance in the latter case has not been undertaken; however, we show in a related paper (Chambers and Healy, 2007) that for any fixed prior with bounded support one can find a symmetric, quasiconcave error distribution and a pair of signals such that the *lower* signal generates a posterior that strictly stochastically dominates the posterior formed from the higher signal. Thus, one can always reverse the Milgrom result with the appropriate choice of error distributions.

We introduce the environment and notation in Section 3. Results appear in Section 4, which is broken into three subsections. The first covers the sufficiency results, the second lists the necessity results, and the third combines these into three characterization theorems.

## 2 Application: Portfolio Selection

We demonstrate our main theorems by applying them to the well-known portfolio allocation problem described by Arrow (1971, Chapter 3).<sup>8</sup> Consider a risk-averse investor allocating a fixed endowment of wealth between a risky asset and a risk-free asset. Initially he makes a portfolio allocation decision based only on his prior beliefs about the risky asset’s return. Then he receives

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<sup>7</sup> This work is followed by Polson (1991), who provides an exact representation of the posterior mean for the case of normally-distributed errors and arbitrary priors.

<sup>8</sup> See also Markowitz (1952, 1958); Pratt (1964); and Tobin (1958) for further discussion of the basic portfolio selection problem.

an informative signal and is asked if he would like to revise his investment decision. In this setting, Observation 5 (below) shows that an outside observer who knows nothing about the shape of the investor’s belief distributions can never predict whether and in what direction the investor will switch his investment strategy after observing the signal. This is true even if the utility function, risk-free return, signal realization *and* prior mean are all public information.

It may seem fairly innocuous to assume, however, that an investor who has chosen to invest a positive amount in the risky asset before seeing a signal will continue to invest some positive amount in it if the signal is, in some sense, favorable. Similarly, one might assume that investors who don’t invest in the risky asset continue not investing after seeing unfavorable signals. Given Observation 5, such assumptions must be placing *some* restriction on the underlying belief distributions, but it is not at all transparent how severe a restriction this implies.

Our general results show that making broad assumptions about the location of the posterior mean with respect to the prior mean and signal realization lead to surprisingly strong restrictions on the shape of the prior distribution. Using a simple equivalence between the location of the posterior mean and the investor’s behavior, we can show that the same strong restrictions are placed on the prior if one assumes that positive signals reinforce positive investments and negative signals reinforce no investment. Thus, models such as these that assume that higher signals are always ‘good news’ and lower signals are always ‘bad news’ may implicitly be ruling out many interesting prior distributions.

To formalize the argument, let the agent’s endowment of wealth be given by  $\omega > 0$  and his utility for money be a differentiable function  $u : \mathbb{R} \rightarrow \mathbb{R}$  with  $u' > 0$  and  $u'' \leq 0$ . The return on the risk-free asset is  $r_0 \geq 1$ , which is known with certainty, and the return on the risky asset is given by the random variable  $X$  with distribution function  $F_X$  and mean  $\mu_X$ . A typical realization of  $X$  is denoted by  $x$ .

The signal shown to the investor is a realization  $z$  of the random variable  $Z := X + \tilde{\varepsilon}$ , where  $\tilde{\varepsilon}$  is a mean-zero error term with distribution  $F_{\tilde{\varepsilon}}$ . Note that the additive representation of the signal is without loss of generality since  $\tilde{\varepsilon}$  may depend arbitrarily on  $X$ ; for this example, however, we do require that for every  $a > 0$  the distribution of  $\tilde{\varepsilon}$  when  $X = \mu_X + a$  be identical to the distribution of  $\tilde{\varepsilon}$  when  $X = \mu_X - a$ . This condition is termed *symmetric dependence* and captures the idea that the signal error around low values of  $X$  should be identical (in distribution) to the signal error around equally high values of  $X$ . Using Bayes’s rule, the investor’s posterior belief about  $X$  upon observing  $Z = z$  is given by the distribution  $F_{X|Z}(\cdot|z)$ .

From a modeling perspective, we will take the prior distribution  $X$  as fixed—

though perhaps unknown—and consider the case where  $F_{\tilde{\varepsilon}}$  might be one of any number of possible error distributions. For example, the signal might represent an early earnings report that, though unbiased, is believed to be more informative by some investors than by others. In the interest of model robustness, one might want to allow for the possibility that *all* independent and uniformly-distributed error distributions are admissible. Thus, assuming that positive signals always reinforce positive investments in the risky asset means assuming this property for all admissible error terms  $\tilde{\varepsilon}$ . The larger the family of admissible error terms, the stronger such assumptions become.

The investor’s choice before receiving a signal is to invest an amount  $y_0 \in \mathbb{R}$  in the risky asset (assuming no credit constraints), with  $\omega - y_0$  being invested in the risk-free asset. Upon observing the signal realization the investor forms updated beliefs and re-solves his optimization problem, resulting in a new investment level  $y_1$ . Formally, the pre-signal investment level is the maximizer of the function

$$V(y_0) := \int u(\omega r_0 + y_0[x - r_0]) dF_X(x)$$

and the post-signal investment (given  $Z = z$ ) maximizes

$$V(y_1|z) := \int u(\omega r_0 + y_1[x - r_0]) dF_{X|Z}(x|z).$$

Here,  $[x - r_0]$  is referred to as the *excess return* of the risky asset. For simplicity, we say that the investor “invests” prior to the signal if  $y_0^* > 0$  and “does not invest” if  $y_0^* \leq 0$ , and that he “invests” after the signal if  $y_1^* > 0$  and “does not invest” if  $y_1^* \leq 0$ .

We now define three different notions of how the investor’s signal might reinforce his prior investment decision.

**Definition 1.** The investor’s decision is *reinforced at the prior mean* if, for all  $r_0$ , he never changes his decision ( $y_1^* = y_0^*$ ) after observing  $Z = E[X]$ .

**Definition 2.** The investor’s decision is *reinforced around the prior mean* if, regardless of  $r_0$ , whenever he has invested ( $y_0^* > 0$ ) and then receives a signal above the prior mean ( $z > E[X]$ ) he continues to invest ( $y_1^* > 0$ ), and whenever he has not invested ( $y_0^* \leq 0$ ) and then receives a signal below the prior mean ( $z < E[X]$ ) he continues not to invest ( $y_1^* \leq 0$ ).

**Definition 3.** The investor’s decision is *reinforced around  $r_0$*  if, regardless of the value of  $r_0$ , whenever he has invested ( $y_0^* > 0$ ) and then receives a signal above the risk-free return ( $z > r_0$ ) he continues to invest ( $y_1^* > 0$ ), and whenever he has not invested ( $y_0^* \leq 0$ ) and then receives a signal below the risk-free return ( $z < r_0$ ) he continues not to invest ( $y_1^* \leq 0$ ).

Although a typical model of portfolio choice may not directly assume these types of reinforcement behaviors, they may assume a certain form of monotonicity in the investor’s behavior (or beliefs) with respect to the signal realization that does imply one or more of these reinforcement behaviors. Furthermore, if the signal realization is public information (such as an earnings report or newspaper story) then whether or not an investor’s decision is reinforced around  $r_0$  is testable using publicly-observed data; if the prior mean is also common knowledge then the other two concepts are testable as well. In either case we could think of the reinforcement behavior as derived from data and then asking what sorts of information structures can rationalize these observed behaviors.

Applying two of our main results gives the following characterizations.

**Theorem 1.** The following are equivalent:

- (1) The investor’s decision is reinforced by the mean signal for all uniformly-distributed error terms  $\tilde{\varepsilon}$ .
- (2) The investor’s decision is reinforced around the prior mean for all uniformly-distributed error terms  $\tilde{\varepsilon}$ .
- (3) The investor’s prior distribution has a symmetric density function.

**Theorem 2.** The investor’s prior has a symmetric and quasiconcave density function if and only if his decision is reinforced around  $r_0$  for all independent, uniformly-distributed error terms  $\tilde{\varepsilon}$ .

Thus, a modeler who assumes that good signals always reinforce a decision to invest (for *all* uniform error distributions) and bad signals always reinforce a decision not to invest is implicitly assuming symmetry and perhaps quasiconcavity of the prior distribution.

We use the remainder of this section to show how all of our general results can be translated into this example, giving the above theorems; the impatient reader may skip ahead to sections 3 and 4 for the general results.

Our three key definitions for this paper are mean reinforcement (MR), updating in the direction of the signal (UDS), and updating toward the signal (UTS), and their formal definitions (Definitions 6–8) are provided below. Each provides structure on how the posterior mean ( $\mu_{X|Z}$ ) relates to the prior mean ( $\mu_X$ ) and the signal realization ( $z$ ). Our general motivation is to identify the restrictions on  $F_X$  and  $F_{\tilde{\varepsilon}}$  that are equivalent to each of these three concepts. In this example, however, we can map the three relatively abstract concepts about belief updating into more concrete restrictions on how the investor will change his investment decision upon observing certain types of signals. This allows us to show an application of how assumed restrictions on behavior can imply strong restrictions on the underlying beliefs, and vice-versa.

In the context of the current example, MR requires that a signal equal to the prior mean not change the investor's expectation about the asset's return. UDS requires that higher signals (relative to the prior mean) lead to higher posterior expectations and lower signals lead to lower expectations, though the relationship between signals and the posterior mean need not be monotonic. UTS strengthens UDS by also requiring that the posterior mean not be any more extreme (relative to the prior mean) than the signal realization. In other words, UDS captures the standard intuition that posterior beliefs should be a weighted average of prior beliefs and the signal realization.

We now show the equivalence between MR, UDS, and UTS and Definitions 1–3, respectively.

**Proposition 3.** An investor's beliefs satisfy MR if and only if his decision is reinforced at the prior mean, his beliefs satisfy UDS if and only if his decision is reinforced around the prior mean, and his beliefs satisfy UTS if and only if his decision is reinforced around  $r_0$ .

This follows from a well-known proposition in the portfolio choice literature.

**Lemma 4.** A risk-averse agent invests a positive (negative) amount in the risky asset if and only if the agent's expected excess return for the risky asset is positive (negative).

*Proof.* The first-order condition for pre-signal maximization at some optimal point  $y_0^*$  is given by

$$V'(y_0^*) = \int [x - r_0] u'(\omega r_0 + y_0^*[x - r_0]) dF_X(x) = 0.$$

The objective function  $V$  is weakly concave in  $y_0$  since  $u'' \leq 0$ , so the first-order condition is also sufficient. Note that at  $y_0 = 0$ ,

$$\begin{aligned} V'(0) &= u'(\omega r_0) \int [x - r_0] dF_X(x) \\ &= u'(\omega r_0) [E[X] - r_0], \end{aligned}$$

so  $V'(0)$  has the same sign as  $E[X] - r_0$ . Thus, if  $E[X] > r_0$  then  $V'(0) > 0$  and so he invests a positive amount in the risky asset. If  $E[X] < r_0$  then the optimal investment quantity is negative. Identical arguments establish that  $y_1^* > 0$  if  $E[X|Z = z] > r_0$  and  $y_1^* < 0$  if  $E[X|Z = z] < r_0$ .  $\square$

*Sketch of the Proof for Proposition 3.* For example, suppose the investor's beliefs satisfy UTS. If he invests before the signal (meaning  $E[X] > r_0$ ) and gets a signal above the risk-free return ( $z > r_0$ ) then we must have  $E[X|Z = z] > r_0$

since, by UTS,  $E[X|Z = z]$  must be between  $E[X]$  and  $z$ . Thus, he continues to invest. If he doesn't invest before the signal ( $E[X] < r_0$ ) and  $z < r_0$  then  $E[X|Z = z] < r_0$  and so he continues not to invest. In the other direction, if his beliefs do not satisfy UTS then there is some  $z$  (say, with  $z \geq E[X]$ ) such that either  $E[X|Z = z] < E[X] \leq z$  or  $E[X] \leq z < E[X|Z = z]$ . In the first, case, if  $r_0$  is such that  $E[X|Z = z] < r_0 < E[X] \leq z$  then the agent invests before the signal, receives a signal above  $r_0$ , but then does not invest. In the second case, if  $r_0$  is such that  $E[X] \leq z < r_0 < E[X|Z = z]$  then the agent does not invest before the signal, receives a signal below  $r_0$ , but then invests. In either case his decision is not reinforced by  $r_0$ . The argument is symmetric for  $z \leq E[X]$ ). Finally, similar arguments can be used for the equivalences between MR and decisions reinforced at the prior mean and between UDS and decisions reinforced around the prior mean.  $\square$

Given Proposition 3, Theorems 1 and 2 then follow directly from Corollaries 15 and 16, respectively, from Subsection 4.3 below.

### 3 Environment and Notation

We study situations in which a decision maker is interested in the realization of some underlying random variable  $X$  but instead observes another random variable  $Z$  that may be informative about  $X$ . Given a particular realization  $z$  of  $Z$  the decision maker forms a posterior expectation about  $X$  (denoted  $E[X|Z = z]$ ) via Bayes's rule. The goal of our study is to identify conditions on the distributions of  $X$  and  $Z$  that guarantee that  $E[X|Z = z]$  lies between  $E[X]$  (the prior mean of  $X$ ) and the signal realization  $z$ , and, conversely, to explore the content of assuming that such a 'betweenness' property—which we call *updating toward the signal* (*UTS*)—holds for a given family of models. We also perform these inquiries for two weaker properties than UTS: *updating in the direction of the signal* (*UDS*), which requires that  $E[X|Z = z] - E[X]$  has the same sign as  $z - E[X]$ , and *mean reinforcement* (*MR*), which requires that  $E[X|Z = z] = E[X]$  whenever  $z = E[X]$ .

Throughout we assume that the real-valued random variables  $X$  and  $Z$  have well-defined density functions  $f_X$  and  $f_Z$ , respectively, with associated cumulative distribution functions  $F_X$  and  $F_Z$ . Random variables are represented by upper-case Roman letters and their realizations by lower-case Roman letters. Conditional densities are defined via Bayes's rule in the usual way, with  $f_{X|Z}(x|z)$  representing the value of the density function of  $X$  at  $x$  conditional on observing  $Z = z$ , for example. In this case,  $f_X$  is referred to as the *prior* density and  $f_{X|Z}$  as the *posterior* density. Means of all random variables are assumed to exist unless otherwise specified.

The two properties of density functions that are central to our paper are symmetry and quasiconcavity<sup>9</sup>:

**Definition 4.** A random variable with density function  $f$  and mean  $\mu$  is **symmetric** if, for every  $a \geq 0$ ,  $f(\mu + a) = f(\mu - a)$ .

**Definition 5.** A random variable with density function  $f$  and mean  $\mu$  is **quasiconcave** if  $|x' - \mu| \leq |x - \mu|$  implies that  $f(x') \geq f(x)$ .

Our results indicate links between quasiconcave and symmetric random variables and UTS, UDS, and MR. Although quasiconcavity and symmetry are strong requirements, they are properties of naturally-occurring processes where small deviations from the mean are more frequent than large deviations.

For our purposes it is convenient to define the random variable  $\tilde{\varepsilon}$  by  $\tilde{\varepsilon} = Z - X$ , so that  $f_{\tilde{\varepsilon}}(\varepsilon|x) = f_{Z|X}(x + \varepsilon|x)$  for each  $x$  and  $\varepsilon$ .<sup>10</sup> Note that  $\tilde{\varepsilon}$  is a function of  $X$  and is more correctly written as  $\tilde{\varepsilon}|X$ , though we drop the dependence on  $X$  in the notation for convenience; the reader should take  $\tilde{\varepsilon}$  as being dependant on  $X$  unless otherwise specified. Throughout this paper we restrict attention to those distributions where  $E[\tilde{\varepsilon}|x] = 0$  for each  $x$ . In this case  $Z = X + \tilde{\varepsilon}$  can be thought of as an unbiased *signal* of  $X$  and the variable  $\tilde{\varepsilon}$  is referred to as a mean-zero *error term*.

We emphasize that writing a signal in the additive form ( $Z = X + \tilde{\varepsilon}$ ) is without loss of generality since  $\tilde{\varepsilon}$  can depend on  $X$ ; for example, a multiplicative error structure of the form  $Z = X\theta$  (as in Lucas, 1972) can be rewritten as an additive structure by defining  $\tilde{\varepsilon} = X\tilde{\theta} - X$ , which clearly depends on  $X$ .

The three key definitions of this paper are as follows:

**Definition 6.** Given a random variable  $X$  and an error term  $\tilde{\varepsilon}$ , the pair  $(X, \tilde{\varepsilon})$  satisfies **mean reinforcement (MR)** if

$$E[X|z = E[X]] = E[X].$$

**Definition 7.** Given a random variable  $X$  and an error term  $\tilde{\varepsilon}$ , the pair  $(X, \tilde{\varepsilon})$  **updates in the direction of the signal** (or, satisfies **UDS**) if for almost every  $z$  in  $\mathbb{R}$  there exists some  $\alpha \geq 0$  such that

$$E[X|z] = \alpha z + (1 - \alpha) E[X]. \tag{1}$$

The order of qualifiers is important in the definition of UDS; because  $\alpha$  can

<sup>9</sup> Quasiconcave density functions are often called “unimodal” in the statistics literature.

<sup>10</sup> We use  $\tilde{\varepsilon}$  to indicate the random variable and  $\varepsilon$  to indicate a particular realization.

vary with  $z$ , UDS does not require that the posterior mean be linear in the signal realization. Instead, UDS only requires that the posterior mean be in the same direction as the signal, relative to the prior mean.

**Definition 8.** Given a random variable  $X$  and an error term  $\tilde{\varepsilon}$ , the pair  $(X, \tilde{\varepsilon})$  **updates toward the signal** (or, satisfies **UTS**) if for almost every  $z$  in  $\mathbb{R}$  there exists some  $\alpha \in [0, 1]$  such that (1) holds.

In the following section we study sufficient conditions on the distributions of  $X$  and  $\tilde{\varepsilon}$  to guarantee that either UTS, UDS, or MR hold for the posterior distribution and, conversely, what properties of  $X$  and  $\tilde{\varepsilon}$  are implied by assuming that UTS, UDS, or MR hold for the posterior when a large family of error terms is considered admissible.

## 4 Results

All of the results below depend on symmetry of the joint distribution  $f_{X,\tilde{\varepsilon}}(x, \varepsilon)$  along the  $x$ -axis in  $(x, \varepsilon)$ -space, so that if  $X$  is mean-zero then  $f_{X,\tilde{\varepsilon}}(x, \varepsilon) = f_{X,\tilde{\varepsilon}}(-x, \varepsilon)$  for almost all  $(x, \varepsilon)$  pairs. Since  $f_{X,\tilde{\varepsilon}}(x, \varepsilon) = f_X(x)f_{\tilde{\varepsilon}}(\varepsilon|x)$ , symmetry is achieved when  $f_X$  is symmetric *and* when  $f_{\tilde{\varepsilon}}(\varepsilon|x) = f_{\tilde{\varepsilon}}(\varepsilon|x)$ . This second condition places the reasonably mild restriction on the structure of the dependence of the error term that the error distribution is identical for two realizations of  $X$  that are equidistant from its mean. Clearly, error terms that are independent of  $X$  satisfy this restriction.

**Definition 9.** An error term  $\tilde{\varepsilon}$  satisfies **symmetric dependence** with respect to  $X$  if

$$f_{\tilde{\varepsilon}}(\varepsilon|X = E[X] + a) = f_{\tilde{\varepsilon}}(\varepsilon|X = E[X] - a)$$

for almost every  $\varepsilon$  and  $a$  in  $\mathbb{R}$ .

Throughout this paper we apply the following assumptions unless otherwise specified.

**A1:** All random variables have well-defined density functions that are continuous on  $\mathbb{R}$  and have finite means.

**A2:**  $E[\tilde{\varepsilon}|x] = 0$  for each realization  $x$  of  $X$ .

**A3:** All error terms  $\tilde{\varepsilon}$  satisfy symmetric dependence.

The first assumption is essentially a regularity condition; although this explicitly rules out random variables whose distribution functions are discontinuous on  $\mathbb{R}$  such as those with two-point distributions of the form  $(a, p; b, 1 - p)$  (where  $a$  occurs with probability  $p$  and  $b$  occurs with probability  $1 - p$ ) or uniform distributions of the form  $U[a, b]$ , such discontinuous distributions can

be arbitrarily approximated by continuous densities and, by continuity, results obtained for those approximations extend to the discontinuous cases.

The second assumption restricts attention to the case of conditionally unbiased signalling structures. Since a Bayesian decision maker must ‘de-bias’ any observations of biased signals by shifting their posterior beliefs appropriately, properties such as UTS will obviously be violated when this de-biasing generates a large enough shift in posterior beliefs; hence, we only consider the case of conditionally unbiased signals.

The third assumption additionally restricts attention to those signalling structures with sufficient symmetry in  $X$ , as described above.

Before deriving our main results, we verify that Bayes’s rule places no restrictions on the location of the posterior mean when no restrictions are placed on the underlying distributions. Thus, restrictions on the pair  $(X, \tilde{\varepsilon})$  are necessary to generate restrictions on the relative locations of the signal realization and the prior and posterior means.

**Observation 5.** For any three real numbers  $\mu$ ,  $z$ , and  $a$  there exist distributions for  $X$  and  $\tilde{\varepsilon}$  such that  $\tilde{\varepsilon}$  is symmetric and independent of  $X$ ,  $E[X] = \mu$  and  $E[X|z] = a$ .

*Proof.* If  $z = a$  simply let  $\tilde{\varepsilon}$  be uniformly distributed with support  $[-1, 1]$  and let  $X$  have a uniform distribution whose support includes  $[a - 1, a + 1]$  so that  $E[X|z = a] = \int_{a-1}^{a+1} x (1/4) dx$ , which equals  $a$ .<sup>11</sup> If  $z \neq a$ , pick any  $f_X$  with  $f_X(a) > 0$  and  $f_X(2z - a) = 0$  and let  $\tilde{\varepsilon}$  have the two-point distribution  $(z - a, 1/2; a - z, 1/2)$  (meaning  $\varepsilon = z - a$  and  $\varepsilon = a - z$  each occur with probability one-half).<sup>12</sup> In this case,

$$E[X|z] = a \frac{f_X(a)}{f_X(a) + f_X(2z - a)} + (2z - a) \frac{f_X(2z - a)}{f_X(a) + f_X(2z - a)},$$

which then reduces to  $E[X|z] = a$ . □

#### 4.1 Sufficient Conditions

We now identify our first set of conditions that provide structure on the posterior mean.

<sup>11</sup> The support of a random variable is the smallest closed set such that the probability measure of its complement is zero.

<sup>12</sup> This  $\tilde{\varepsilon}$  can be approximated using a continuous density function to obtain the same result.

**Proposition 6.** If  $X$  and  $\tilde{\varepsilon}$  are symmetric then the pair  $(X, \tilde{\varepsilon})$  satisfies mean reinforcement (MR).

*Proof.* Without loss of generality assume that  $E[X] = 0$ . Letting  $z = 0$ , we now wish to show that  $E[X|0] = 0$ . To prove statements such as this we rely on the following lemma:

**Lemma 7.** If  $X$  is symmetric and  $\tilde{\varepsilon}$  satisfies symmetric dependence then  $E[X|z] \stackrel{\geq}{\leq} 0$  if and only if

$$\int_0^\infty x [f_{\tilde{\varepsilon}}(z - x|x) - f_{\tilde{\varepsilon}}(z + x|x)] f_X(x) dx \stackrel{\geq}{\leq} 0. \quad (2)$$

*Proof.* Suppose  $E[X|z] \geq 0$  for some  $z$  (the proofs for  $E[X|z] = 0$  and  $E[X|z] \leq 0$  are identical). Then

$$\int_{-\infty}^\infty x f_{X|Z}(x|z) dx \geq 0.$$

By splitting the integral at zero, this becomes

$$\int_0^\infty x f_{X|Z}(x|z) dx \geq \int_{-\infty}^0 -x f_{X|Z}(x|z) dx,$$

or, by a change of variables and rearranging,

$$\int_0^\infty x [f_{X|Z}(x|z) - f_{X|Z}(-x|z)] dx \geq 0.$$

By Bayes's rule  $f_{X|Z}(x|z)$  is proportional to  $f_{Z|X}(z|x) f_X(x)$ , so the above inequality becomes

$$\int_0^\infty x [f_{\tilde{\varepsilon}}(z - x|x) f_X(x) - f_{\tilde{\varepsilon}}(z + x|x) f_X(-x)] dx \geq 0. \quad (3)$$

Symmetry of  $X$  implies that  $f_X(x) = f_X(-x)$  and symmetric dependence implies that  $f_{\tilde{\varepsilon}}(\varepsilon|x) = f_{\tilde{\varepsilon}}(\varepsilon|-x)$  for any  $x$  and  $\varepsilon$ . Therefore, equations 2 and 3 are equivalent.  $\square$

Thus, proving  $E[X|0] = 0$  is equivalent to showing that

$$\int_0^\infty x [f_{\tilde{\varepsilon}}(0 - x|x) - f_{\tilde{\varepsilon}}(0 + x|x)] f_X(x) dx = 0.$$

But the bracketed term equals zero almost everywhere since  $\tilde{\varepsilon}$  is symmetric, so the claim follows immediately.  $\square$

Proposition 6 demonstrates that symmetry of both the prior and the error

distribution is sufficient to guarantee mean reinforcement. The following example shows that these conditions are not sufficient, however, for the stronger condition of updating in the direction of the signal (UDS).

**Example 1.** Consider an  $X$  that is uniformly distributed over  $[-2, 2]$  and an error term  $\tilde{\varepsilon}$  with the symmetric two-point distribution  $(-10, 1/2; 10, 1/2)$ . If the signal realization is  $z = 9$  then it is known with certainty that  $x = -1$  and  $\varepsilon = 10$ . UDS requires  $E[X|z = 9] > 0$  but here  $E[X|z = 9] = -1$ , so UDS fails.

In order to guarantee UDS a stronger condition is needed. From Example 1 it is clear that quasiconcavity of the prior is not sufficient. Instead, we find that quasiconcavity of the *error* distribution is sufficient for UDS, even when the prior is symmetric but not quasiconcave.

**Proposition 8.** If  $X$  is symmetric and  $\tilde{\varepsilon}$  is symmetric and quasiconcave then the pair  $(X, \tilde{\varepsilon})$  satisfies UDS.

*Proof.* Without loss of generality, let  $E[X] = 0$ . Assume first that  $z \geq 0$ , in which case we wish to show that  $E[X|z] \geq 0$ . Using Lemma 7, this is equivalent to showing that

$$\int_0^\infty x [f_{\tilde{\varepsilon}}(z - x|x) - f_{\tilde{\varepsilon}}(z + x|x)] f_X(x) dx \geq 0. \quad (4)$$

Using the symmetry of  $\tilde{\varepsilon}$ , the bracketed term can be rewritten as

$$[f_{\tilde{\varepsilon}}(|z - x| |x) - f_{\tilde{\varepsilon}}(z + x|x)].$$

It must be that  $|z - x| \leq z + x$  for all non-negative  $x$ , so quasiconcavity of  $\tilde{\varepsilon}$  guarantees that  $f_{\tilde{\varepsilon}}(|z - x| |x) \geq f_{\tilde{\varepsilon}}(z + x|x)$ . Thus, the bracketed term in equation (4) is non-negative almost everywhere, proving the result for  $z \geq 0$ ; the proof for  $z \leq 0$  is symmetric.  $\square$

We now consider the stronger condition of updating *toward* the signal (UTS). The following example shows that symmetry of the prior and symmetry and quasiconcavity of the error distribution—which were sufficient for UDS—are not sufficient for UTS.

**Example 2.** In this example we simply switch the role of the two distributions in Example 1. Specifically, let  $X$  have the symmetric two-point distribution  $(-10, 1/2; 10, 1/2)$  and let  $\tilde{\varepsilon}$  have the (symmetric and quasiconcave) uniform distribution over  $[-2, 2]$ . Now when  $z = 9$  it is known with certainty that  $x = 10$  and  $\varepsilon = -1$ . UTS requires that  $E[X|z = 9] \in [0, 9]$  but  $E[X|z = 9] = 10$ , so UTS fails.

Our final sufficiency result shows that UTS is satisfied when *both* the error distribution and the prior distribution are symmetric and quasiconcave and, in addition, the error term is independent of the prior. It is well known that if the prior and error term are independent and normally distributed then the pair satisfies UTS; this fact can now be derived as a corollary of the following result.

**Proposition 9.** If  $X$  is symmetric and quasiconcave and  $\tilde{\varepsilon}$  is symmetric, quasiconcave, and independent of  $X$  then the pair  $(X, \tilde{\varepsilon})$  satisfies UTS.

*Proof.* Without loss of generality let  $E[X] = 0$  and consider the case where  $z \geq 0$ . By Proposition 8 we know that UDS is satisfied, so  $E[X|z] \geq 0$ . It remains to show that  $E[X|z] \leq z$ .

Note that since  $Z = X + \tilde{\varepsilon}$ ,  $E[X|z] \leq z$  if and only if  $E[\tilde{\varepsilon}|z] \geq 0$ , or

$$\int_{-\infty}^{\infty} \varepsilon f_{\tilde{\varepsilon}|Z}(\varepsilon|z) d\varepsilon \geq 0.$$

Using the same methods as in the proof of Lemma 7, we break the integral at zero, use a change of variables, and apply Bayes's rule to show that this expression is equivalent to

$$\int_0^{\infty} \varepsilon [f_{\tilde{\varepsilon}}(\varepsilon|z - \varepsilon) f_X(z - \varepsilon) - f_{\tilde{\varepsilon}}(\varepsilon|z + \varepsilon) f_X(z + \varepsilon)] d\varepsilon \geq 0. \quad (5)$$

Using independence of  $\tilde{\varepsilon}$ , this can be rewritten as

$$\int_0^{\infty} \varepsilon f_{\tilde{\varepsilon}}(\varepsilon) [f_X(z - \varepsilon) - f_X(z + \varepsilon)] d\varepsilon \geq 0.$$

Since  $|z - \varepsilon| \leq z + \varepsilon$  for non-negative  $\varepsilon$ , symmetry and quasiconcavity of  $f_X$  imply that the bracketed term is non-negative almost everywhere. Thus, the integral is non-negative, proving the result. The proof for  $z \leq 0$  is symmetric.  $\square$

## 4.2 Necessary Conditions

We now explore conditions on the prior and error distributions that must be true of the pair satisfies MR, UDS, and UTS. The following example shows, however, that no combination of symmetry and quasiconcavity is necessary for MR, UDS, or UTS.

**Example 3.** Let  $X$  and  $\tilde{\varepsilon}$  both have the two-point distribution  $(4, 1/3; -2, 2/3)$ . This mean-zero distribution is neither symmetric nor quasiconcave, though  $\tilde{\varepsilon}$  is

independent of  $X$  and therefore satisfies symmetric dependence. The three possible signal realizations are  $z = 8$ ,  $z = 2$ , and  $z = -4$ . If  $z \in \{8, -4\}$  then  $x$  is known with certainty. In particular,  $E[X|z = 8] = 4$  and  $E[X|z = -4] = -2$ . If  $z = 2$  is observed then  $x = 4$  and  $x = -2$  are equally likely; thus,  $E[X|z = 2] = 1$ . In all three cases  $E[X|z] = z/2$ , so UDS and UTS are satisfied and MR is trivially satisfied.

Given this difficulty in identifying necessary conditions for any particular pair  $(X, \varepsilon)$ , we follow Milgrom (1981) and ask what conditions MR, UTS, and UDS would imply on one of the distributions given freedom in the specification of the other. Specifically, we take the viewpoint of a modeler who fixes a prior distribution but, in the interest of model robustness, is unwilling to commit to a particular error distribution. For example, she may select a normally-distributed prior but allow for any symmetric and quasiconcave error distribution. In this setting we ask what the restrictions on the prior must be if MR, UDS, or UTS is assumed to hold for *all* such error distributions.

We focus on two cases of interest: first, when the modeler requires MR, UDS, or UTS to hold for all symmetric, independent, two-point error distributions (or, by extension, any larger family of error distributions that includes all symmetric, two-point distributions) and, second, when the modeler requires MR, UDS, or UTS to hold for all independent, uniformly-distributed error terms (or any larger family).

Our interpretation of these two families of error distributions is that any sufficiently broad family of error terms that includes symmetric (but not necessarily quasiconcave) distributions will include all symmetric, independent, two-point error distributions as a subset, and any sufficiently broad family of error terms that includes symmetric and quasiconcave distributions will include all independent, uniformly-distributed error terms as a subset. The results of this section explore the freedom in the choice of the prior that can be obtained when one requires MR, UDS, or UTS while allowing for such flexibility in the specification of the error terms.

Proposition 6 showed that symmetry of both the error and prior distribution is sufficient for mean reinforcement (MR). The following provides a rough converse to this result: if all symmetric two-point distributions are admissible, then requiring MR for all admissible error terms implies that the prior distribution must be symmetric.

**Proposition 10.** If  $X$  is such that  $(X, \varepsilon)$  satisfies mean reinforcement (MR) for all two-point error terms  $\tilde{\varepsilon}$  then  $X$  is symmetric.

*Proof.* We prove the contrapositive statement, which says that if  $X$  is not symmetric then there is some two-point error term for which MR fails.

Assume without loss of generality that  $E[X] = 0$ . Since  $X$  is not symmetric there is some  $y > 0$  such that  $f_X(y) \neq f_X(-y)$ . Consider the error term  $\tilde{\varepsilon}$  with the two-point distribution  $(-y, 1/2; y, 1/2)$  (or, to avoid difficulties associated with discontinuous distributions, consider a sequence of continuous error terms whose distributions converge to the two-point distribution  $(-y, 1/2; y, 1/2)$  in the weak topology and consider the limit of their posteriors). The posterior mean given  $z = 0$  is then

$$E[X|z = 0] = \frac{yf_X(y) - yf_X(-y)}{f_X(y) + f_X(-y)},$$

which cannot equal zero (the prior mean) since  $f_X(y) \neq f_X(-y)$ . Thus, MR fails.  $\square$

With a broad enough class of symmetric error terms, the above result shows that mean reinforcement implies symmetry of the prior. We now demonstrate that one *cannot* use the stronger condition of UDS if this same family of error terms is admissible.

**Proposition 11.** For all  $X$  there exists some symmetric error term  $\tilde{\varepsilon}$  with a two-point error distribution such that the pair  $(X, \tilde{\varepsilon})$  does not satisfy UDS.

*Proof.* Suppose, by way of contradiction, that  $X$  satisfies UDS for all symmetric  $\tilde{\varepsilon}$  with two-point distributions. Pick any pair  $(x_1, x_2)$  with  $x_2 > x_1 > 0$  and consider the error term (or sequence of continuous error terms converging to)

$$\tilde{\varepsilon}_{(x_1, x_2)} \sim \left( -\frac{x_1 + x_2}{2}, \frac{1}{2}; \frac{x_1 + x_2}{2}, \frac{1}{2} \right)$$

and the signal  $z = (x_2 - x_1)/2$  when  $Z = X + \tilde{\varepsilon}_{(x_1, x_2)}$ . In this case, the posterior on  $X$  given  $z$  has the two-point distribution

$$\left( -x_1, \frac{f_X(-x_1)}{f_X(-x_1) + f_X(x_2)}; x_2, \frac{f_X(x_2)}{f_X(-x_1) + f_X(x_2)} \right).$$

Thus,  $E[X|z]$  is proportional to  $-x_1 f_X(-x_1) + x_2 f_X(x_2)$ , which must be non-negative since  $X$  satisfies UDS. Therefore,  $x_2 f_X(x_2) \geq x_1 f_X(-x_1)$ . Since UDS implies MR, we know by Proposition 10 that  $X$  must be symmetric, so we have that

$$x_2 f_X(x_2) \geq x_1 f_X(x_1).$$

This is true for arbitrary  $x_1 > x_2 > 0$ , so  $x f_X(x)$  is increasing in  $x$  for  $x > 0$ . But then  $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$  does not exist unless  $f_X$  is zero almost everywhere. This is a contradiction since UDS requires that  $X$  have a finite mean.  $\square$

**Remark.** Morris and Shin (2002) assume an ‘improper’ uniform prior over

the entire real line. This prior satisfies MR, UDS, and UTS for *all* error terms—apparently contradicting Proposition 11—since the posterior mean always equals the signal realization. This prior is ruled out, however, by our assumption that  $X$  has a well-defined and finite mean. If the improper prior were considered admissible then Proposition 11 would be violated but all other results would hold without modification.

The above impossibility result implies that UDS and UTS cannot be satisfied if all symmetric error distributions are admissible. The sufficiency results, however, indicate that quasiconcavity also plays a role in both UDS and UTS, so we proceed by adding quasiconcavity to the admissible family of error terms. This is done by considering the family of all uniformly-distributed error terms. We now show that in this case symmetry of  $X$  is not only implied by UDS, but that it is implied by the weaker concept of mean reinforcement as well.

**Proposition 12.** If  $X$  is such that  $(X, \tilde{\varepsilon})$  satisfies mean reinforcement (MR) for all uniformly-distributed error terms  $\tilde{\varepsilon}$  then  $X$  is symmetric.

*Proof.* Suppose that  $(X, \tilde{\varepsilon})$  satisfies MR for every uniformly-distributed error term  $\tilde{\varepsilon}$ . Pick any  $y > 0$  and choose  $\tilde{\varepsilon}_y$  to have the uniform distribution over  $[-y, y]$ . In this case the posterior mean (or, more precisely, the limit of the sequence of posterior means) given  $z = 0$  is proportional to  $\int_{-y}^y x f_X(x) dx$ . If  $(X, \tilde{\varepsilon}_y)$  satisfies MR for every  $y > 0$ , then

$$\int_{-y}^y x f_X(x) dx = 0 \tag{6}$$

for almost every  $y > 0$ . Differentiating this expression with respect to  $y$  establishes that

$$y f_X(y) = y f_X(-y)$$

or

$$f_X(y) = f_X(-y)$$

for almost every  $y > 0$ . Thus,  $X$  is symmetric.  $\square$

Finally, we saw from the sufficiency results that quasiconcavity of the prior can be used to guarantee UTS, so we demonstrate now that UTS implies that the prior should also be quasiconcave when all uniform distributions are admissible.

**Proposition 13.** If  $X$  is such that  $(X, \tilde{\varepsilon})$  satisfies UTS for all uniformly-distributed error terms  $\tilde{\varepsilon}$  then  $X$  is symmetric and quasiconcave.

*Proof.* Proposition 12 and the fact that UTS implies MR gives the result when  $X$  is not symmetric; we now demonstrate a failure of UTS when  $X$  is symmetric but not quasiconcave.

Let  $E[X] = 0$  (without loss of generality) and consider any  $z \geq 0$ . UTS implies that  $E[X|z] \leq z$ , or equivalently that  $E[\tilde{\varepsilon}|z] \geq 0$ . Consider the error term  $\tilde{\varepsilon}_y$  with a uniform distribution over  $[-y, y]$ . Then  $E[\tilde{\varepsilon}|z] \geq 0$  if and only if

$$\int_{-y}^y \varepsilon f_X(z - \varepsilon) d\varepsilon \geq 0.$$

By breaking this integral at  $y = 0$  and applying a change of variables the expression can be rewritten as

$$\int_0^y \varepsilon [f_X(z - \varepsilon) - f_X(z + \varepsilon)] d\varepsilon \geq 0. \quad (7)$$

Since  $f_X$  is not quasiconcave then, by symmetry, there is a pair of values  $x'' > x' \geq 0$  such that  $f_X(x'') > f_X(x')$ . Fix some  $\alpha \in (f_X(x'), f_X(x''))$ . Let  $\hat{x} = \sup\{x \in [x', x''] : f_X(x) < \alpha\}$  so that  $f_X(\hat{x}) = \alpha$  by continuity. Now let  $x^* = \inf\{x \geq \hat{x} : f_X(x) > \alpha\}$  so that  $f_X(x^*) = \alpha$  as well, though it may be the case that  $x^* > \hat{x}$ . Since  $\alpha > f_X(x')$  it must be that  $\hat{x} > x' \geq 0$ , and so  $x^* > 0$ . Thus there is some  $\delta > 0$  small enough such that  $x^* - \delta > 0$  and, for each  $\varepsilon \in (0, \delta)$ ,  $f_X(x^* - \varepsilon) \leq \alpha$  and  $f_X(x^* + \varepsilon) > \alpha$ . We have therefore established an interval  $(x^* - \delta, x^* + \delta)$  with  $f_X \leq \alpha$  on the lower half of the interval and  $f_X > \alpha$  on the upper half. Thus,  $f_X(x^* + \varepsilon) > f_X(x^* - \varepsilon)$  for all  $\varepsilon \in (0, \delta)$ . But if we consider the case where the signal realization is  $z = x^*$  and the error term is uniformly distributed on  $[-y, y]$  with  $y = \delta$  then UTS implies equation 7, which contradicts the fact that  $f_X(z + \varepsilon) > f_X(z - \varepsilon)$  for almost every  $\varepsilon$  in  $[0, y]$ . Thus, UTS fails. The proof for  $z \leq 0$  is symmetric.  $\square$

### 4.3 Characterizations

All sufficiency and necessity results are summarized in Table I. Combining each sufficiency result with the appropriate necessity result allows us to state three characterization theorems that roughly summarize the findings throughout the paper.

The first characterization highlights the link between mean reinforcement and symmetry of the prior and error distributions.

**Corollary 14.**  $X$  is symmetric if and only if  $(X, \tilde{\varepsilon})$  satisfies MR for every symmetric, two-point error term  $\tilde{\varepsilon}$ .

*Proof.* Follows immediately from Propositions 6 and 10.  $\square$

The second characterization is perhaps the most surprising; we show that MR is exactly as strong as UDS when the family of admissible error terms is a

Result	Error Terms	Prior	Condition
Prop. 6	Sym	Sym $\Rightarrow$	MR
Ex. 1	Sym	Sym $\not\Rightarrow$	UDS
Prop. 8	Sym QC	Sym $\Rightarrow$	UDS
Ex. 2	Sym QC	Sym $\not\Rightarrow$	UTS
Prop. 9	Sym QC Ind	Sym QC $\Rightarrow$	UTS
Prop. 10	All Two-Point Errors	Sym $\Leftarrow$	MR
Prop. 11	All Two-Point Errors	$\not\Leftarrow$	UDS
Prop. 12	All Uniform Errors	Sym $\Leftarrow$	MR
Prop. 13	All Uniform Errors	Sym QC $\Leftarrow$	UTS

**Key:** Sym = symmetric, QC = quasiconcave, and Ind = independent.

Table I  
Summary of results.

broad enough family of symmetric and quasiconcave errors.

**Corollary 15.** The following are equivalent:

- (1)  $X$  is such that  $(X, \tilde{\varepsilon})$  satisfies MR for all uniformly-distributed error terms  $\tilde{\varepsilon}$ .
- (2)  $X$  is such that  $(X, \tilde{\varepsilon})$  satisfies UDS for all uniformly-distributed error terms  $\tilde{\varepsilon}$ .
- (3)  $X$  is symmetric.

*Proof.* That (2) implies (1) follows from a simple continuity argument on  $E[X|z]$  at  $z = 0$ . We know that (1) implies (3) by the contrapositive of Proposition 12, and we know that (3) implies (2) by Proposition 8 since all uniform distributions are symmetric and quasiconcave. Thus, the three statements are equivalent.  $\square$

Finally, we find an equivalence between priors that satisfy UTS for all independent, uniformly-distributed error terms and priors whose distributions are symmetric and quasiconcave.

**Corollary 16.**  $X$  is symmetric and quasiconcave if and only if  $X$  is such that  $(X, \tilde{\varepsilon})$  satisfies UTS for all independent, uniformly-distributed error terms  $\tilde{\varepsilon}$ .

*Proof.* Follows immediately from Propositions 9 and 13.  $\square$

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